Representation theory from lie algebra perspective

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§1 Homorphism theorems

Definition 1.1. Let L be a Lie algebra and let V and W be $L-$modules. An $L-$module homomorphism or Lie homomorphism from V to W is a linear map $\theta : V \to W$ such that $\theta(x \cdot v) = x \cdot \theta(v)$ for all $v \in V$ and $x \in L$.

An isomorphism is a bijective $L-$module homomorphism.

Remark 1.2. This makes sense because $x \cdot v \in V$ and $\theta v \in W$. So $x \cdot \theta(v)$ is also in W. And it is using that it is $L-$module.

§2 Schur's lemma

Theorem 2.1. Let L be a complex lie algebra and V be a finite dimensional simple $L-$module where $\theta : V \to V$ is an $L-$module homomorphism. Then $\theta = \lambda \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.

Proof. Since we are working in $\mathbb{C}$, let $\lambda \in \mathbb{C}$ be an eigenvalue of θ. Let $v \in V$ be the corresponding eigenvector. So

$$\theta(v) = \lambda v \implies v \in \mathrm{Null}(\theta - \lambda \mathrm{id}_V) \implies \{0\} = \mathrm{Null}(\theta - \lambda \mathrm{id}_V) \subset V \text{ is a submodule}$$

$$V = \mathrm{Null}(\theta - \lambda \mathrm{id}_V) \implies \forall u \in V, \theta(u) = \lambda(u) \implies \theta = \lambda \mathrm{id}_V.$$

□

Lemma 2.2. Let L be a complex lie algebra and abelian. And V be a simple finite dimensional module. Then $dim(V) = 1$.

Proof. We define $\theta_x : V \to V$ and $\theta_x(v) = x \cdot v$. Note that $\theta_x : V \to V$ is an $L-$module homorphsim. Note that

$$y \cdot \theta_x(v) = y(x \cdot v) = x \cdot (y \cdot v) - [x, y] \cdot v = \theta_x(y \cdot v).$$

Since L is complex and θ_x is $V \to V \implies \theta_x = \lambda_x \mathrm{id}_V$. So $x \cdot v = \theta_x v = \lambda_x v \implies \mathrm{span} v \subset V$ is submodule. □

Cartan's criteria

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§1 Cartan's criteria for solvability

Theorem 1.1. Suppose L is a complex solvable lie algebra. Suppose L is a lie algebra. Then

$$L \text{ is solvable } \iff \forall x, y, z \in L, k([x, y], z) = 0.$$

Proof. Suppose L is solvable. Then consider $ad : L \to gl(L)$. Note that $im(ad)$ is a quotient and homomorphism of L, hence solvable. So there exists a basis of L such that $\forall w \in L$, ad_w is upper triangular matrix. Hence $[ad_x, ad_y]$ is upper triangular matrix. Hence $[ad_x, ad_y]ad_z$ is is upper triangular matrix. So $Tr([ad_x, ad_y]ad_z) = 0 \implies k([x, y], z) = 0$.
Suppose $\forall x, y, z \in L$, we have $k([x, y], z) = 0$. We will show that L' is nilpotent. Note that $w = [x, y] \in L' \implies k(w, z) = 0$. We know that by jordan decomposition, $ad_w = (ad_w)_d + (ad_w)_n = W_d + W - n$. But note that $tr(W \cdot \overline{W_d}) = tr(W_d \cdot \overline{W_d}) = |\lambda_1|^2 + \cdots + |\lambda_m|^2$ and λ_i is eigenvalue of W. But $tr(W \cdot \overline{W_d}) = tr(ad_{[x,y]} \cdot ad_{\overline{w}_d}) = 0$. So the eigenvalues are 0. So L' is nilpotent. So L is solvable. □

§2 Cartan's criteria for semi simplicity

Definition 2.1. $W^\perp = \{x \in W | k(x, w) = 0 \forall w \in W\}$ where k is the killing form and W is subspace of lie algebra L.

Theorem 2.2. If I is ideal of L then $I^\perp$ is ideal of L.

Proof. Let $x \in L$ and $j \in I^\perp$. We want to show $[j, x] \in I^\perp$. We want to show

$$tr(ad_{[j,x]} \cdot ad_i) = 0 \forall i \in I.$$

But we know that

$$tr([a, b]c) = tr(a[b, c]).$$

So

$$tr(ad_{[j,x]} \cdot ad_i) = tr(ad_j \cdot ad_{[x,i]}) \text{ but } [x, i] \in I \implies tr(ad_j \cdot ad_{[x,i]}) = 0.$$

□

Theorem 2.3. Suppose L is a lie algebra over $\mathbb{C}$. Then L is semi-simple $\iff$ k is non-degenerate. That is if $x \neq 0 \implies k(x, x) \neq 0$.

Proof. Suppose L is semi-simple.So rad L is 0. So there is no solvable ideal of L. Note that $L \subset L$ is an ideal. Now take $x, y, z \in L^{\perp} \implies [x, y] \in (L^{\perp})'$. Note that $k([x, y], z) = 0$ as $[x, y] \in L$. So $L^{\perp}$ is solvable (by the criterion) and ideal. So $L^{\perp} = 0$. Hence k is non-degenarate.
Other way: Say L is not simple $\implies$ it has solvable ideals. Take $0 \neq I \subset L$ a solvable ideal $\implies \exists N \in \mathbb{N}$ such that $I^{(N)} = 0$ and $I^{(N-1)} \neq 0$. Let $A = I^{(N-1)}$. Now $\forall y \in I$ we have

$$ad_a ad_x ad_a(y) = [a, [x, [a, y]]] = 0 \implies (ad_a ad_x)^2 = 0$$

$$\implies ad_a \cdot ad_x \text{ is nilpotent } \implies tr(ad_a \cdot ad_x) = 0 \implies k(a, x) = 0 \implies k(a, a) = 0.$$

□

§3 Why is it called semi-simple

Now, we will understand why it is called semi-simple!

> **Theorem 3.1.** Let L is a lie algebra over $\mathbb{C}$. Then L is semisimple $\implies \exists$ simple ideals $L_1, \ldots, L_n \subset L$ such that $L = L_1 \bigoplus \cdots \bigoplus L_n$.

Proof. We do induction on dim L. Base case: dim $L = 1 \implies L$ is simple. Say for all lie algebras of dim $< n$, the statement holds and dim $L = m$. Let $I \subset L$ be ideal of L and minimal dimension. If $I = L$ then L is simple. If $I \neq L$ then we have the following claim.

> **Claim 3.2.**
> $$L = I \bigoplus I^{\perp}, I, I^{\perp} \text{ are semi-simple }.$$

Proof. Note that

$$x \neq 0 \in I \cap I^{\perp} \implies k(x, x) = 0 \implies k \text{ is degenerate } \implies x = 0.$$

Note that $I, I^{\perp}$ commute.

$$[x, w] \in I, I^{\perp} \implies [x, w] = 0 \forall x, w \in I, I^{\perp}.$$

Note that $L = I + I^{\perp}$ as $I \to I \to \mathbb{C}$ is isomorphism. So $V \to I \to \mathbb{C}$ is surjective and kernel is $I^{\perp}$. And dimensions follow! They are semi-simple, because suppose $J \subset I$ is a solvable ideal. Then

$$[J, I^{\perp}] \subset [I, I^{\perp}] = 0 \implies J \subset I^{\perp} \text{ and solvable.}$$

Not possible. So both are semi-simple. □

Now use induction on $I^{\perp}$.
Other direction: Suppose

$$L = L_1 \bigoplus \cdots \bigoplus L_n.$$

Let $I = radL$. Let $I_k = [I, L_k]$. Note that $I_k \subset L_k$ a solvable ideal. So $I_k = 0$. So

$$[I, L] = [I, L_1 \bigoplus \ldots L_n] \subset I_1 \bigoplus \cdots \bigoplus I_n = 0$$

$$\implies I \subset Z(L) \subset Z(L_1) \bigoplus \cdots \bigoplus Z(L_n) = 0.$$

□

Weights, Invariance lemma, Engel's theorem and Lie's theorem

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We are dealing with lie algebras

§1 Weights

Definition 1.1. A weight for a lie subalgebra A of gl(V) is a linear map $\lambda : A \to F$ such that

$$V_\lambda = \{v \in V : a(v) = \lambda(a)v \forall a \in A\}$$

Note that this is the generalisation of eigenvectors.
Note that V_λ forms a vector subspace of V as if $v, w \in V_\lambda$ then

$$a(\alpha v + \beta w) = a(\alpha v) + a(\beta w) = \alpha a(v) + \beta a(w)$$

$$\alpha\lambda(a)v + \beta\lambda(a)w = \lambda(a)(\alpha v + \beta w).$$

Problem 1.2. If $a, b : V \to V$ are commuting linear transformations and W is the kernel of a, then W is $b-$invariant.

Proof. Let $w \in W$, then

$$a(bw) = b(aw) = 0 \implies bw \in W.$$

□

§2 The invariance lemma

Lemma 2.1. Suppose that A is an ideal of a Lie subalgebra L of $\mathrm{gl}(V)$. Let $W = \{v \in V : a(v) = 0 \forall a \in A\}$. Then W is an L-invariant subspace of V .

Here we use the famous trick that $ax = xa - [a, x]$

Proof. Let $w \in W$ and $x \in L$. We have to show that $a(xw) = 0 \forall a \in A$. Note that

$$a(xw) = x(aw) + [a, x](w) = 0.$$

□

Theorem 2.2 (Invariance lemma)**.** Assume that F has characteristic zero. Let L be a Lie subalgebra of $\mathrm{gl}(V)$ and let A be an ideal of L. Let $\lambda : A \to F$ be a weight of A. The associated weight space

$$V_\lambda = \{v \in V : av = \lambda(a)v \forall a \in A\}$$

is an L-invariant subspace of V .

Proof. If $y \in L$ and $w \in V_\lambda$ then $y(w) \in V_\lambda$. That is, to show that $\forall a \in A, a(y(w)) = \lambda(a)(y(w))$.
Again, using the above trick, we get

$$a(yw) = y(aw) + [a, y](w) = y(\lambda(a)w) + \lambda([a, y])(w) = \lambda(a)yw + \lambda([a, y])(w).$$

So it is enough to show $\lambda([a, y])(w) = 0$.
Consider $U = \mathrm{span}\{w, y(w), \dots, y^{m-1}(w)\}$ such that they are linearly independent and hence $w, y(w), \dots, y^{m-1}(w)$ are basis of U.

Claim 2.3. Let $z \in A$, then wrt the basis $w, y(w), \dots, y^{m-1}(w)$, z is represented as an Upper triangular matrix with diagonal entries $\lambda(z)$.

Proof. We prove it by induction on columns (from left to right). For our base case, it is true as $zv = \lambda(z)v$. For yw, note that

$$z(yw) = y(zw) + [z, y]w = \lambda(z)(yw) + \lambda[z, y](w)$$

which is the second column.
Say it is true till k th column. Hence,

$$z(y^{r-1}w) = \lambda(z)(y^{r-1}w) + u, u \in \ \mathrm{span}\{w, y(w), \dots, y^{r-2}(w)\}.$$

So $k + 1$th colum will be

$$z(y^r w) = zy(y^{r-1}w) = (yz + [z, y])(y^{r-1}w) = yzy^{r-1}w + [z, y]y^{r-1}w$$
$$= \lambda(z)(y^r w) + yu + [z, y]y^{r-1}w$$

Since $[z, y] \in A$, we get $[z, y]y^{r-1}w = \lambda([z, y])y^{r-1}w \in\in \ \mathrm{span}\{w, y(w), \dots, y^{r-1}(w)\}$. So induction works. ☐

Now take $z = [a, y]$. Note that the trace of the matrix of z acting on U is $m\lambda(z)$. U is invariant under the action of $a \in A$, and U is y-invariant by construction. Note that trace is 0. So $\lambda(z) = 0$ as char is 0. So done. We proved it. ☐

Problem 2.4. Let $x, y : V \to V$ be linear maps from a complex vector space V to itself. Suppose that x and y both commute with $[x, y]$. Then $[x, y]$ is a nilpotent map.

Proof. Note that if λ be an eigen value of the linear map $[x, y]$. Let $W = \{v \in V : [x, y]v = \lambda v\}$ be the eigenspace. Note that it is a subspace of V.
Let L be the lie algebra of gl(V) spanned by x, y, and $[x, y]$. Note that span $\{[x, y]\}$ is an ideal of L. So the invariance lemma implies that W is invariant under L. So it is invariant under x and y.
Pick any basis of W and let X and Y be the matrices of x and y wrt the basis. So $[x, y] = XY - YX$. But every element of W is an eigenvector of $[x, y]$ with eigenvalue λ.

Claim 2.5. $XY - YX$ is a scalar matricx with scalar λ wrt the basis

However, taking trace, gives us

$$0 = \lambda \text{dim} W \implies \lambda = 0.$$

Claim 2.6. If a linear operator $\phi : V \to V$ on a vector space is nilpotent, then its only eigenvalue is 0 And it similarly if only eigenvalues are 0 then it is nilpotent.

Proof. Let A be a nilpotent, so $A^n = 0$ for some n. Now let v be an eigenvector: $Av = \lambda v$ for some scalar λ. Now we get

$$0 = A^n v = \lambda^n v \implies \lambda = 0.$$

Other direction: Suppose that all the eigenvalues of the matrix A are zero. Then the characteristic polynomial of the matrix A is

$$p(t) = det(A - tI) = \pm t^n \implies 0 = p(A) = \pm A^n \implies A \text{ is nilpotent }.$$

□

So using the above claim, we are done. □

§3 Engel's theorem

We will try to prove Engel's theorem. But what does it say? But first we state a common result in linear algebra.

Theorem 3.1. Let V be a n dim vector space. And let $x : V \to V$ be a nilpotent map. Then there exists a basis of V is which x is a Upper triangular matrix.

Proof. Since x is nilpotent, $\exists N \in \mathbb{N}$ such that $x^N = 0$. Now $0 \neq v \in V \implies x^N(v) = 0$.
Let m be smallest m such that $w = x^{m-1}v \neq 0$. So $x(w) = 0 \implies w \in \text{Null}(x)$.
If $n = 1$. Then $V = \text{span}(w)$ as $w \neq 0$. So the transformation x's matrix wrt w is $[0]$.
Say the statement is true for any k dimensional vector space. We will show that it is true for any $k + 1$ dimensional vector space.
Let $W = \text{span} w$ which is subspace of V. So V/W is k dimensional.
Note that $\overline{x}$ is nilpotent as $\overline{x} = x + W$ and

$$(\overline{x})^n = (x + W)^n = x^n + W.$$

And $x^n = 0 \in V, W \subseteq V$. So $(\overline{x})^n = 0 + W$. Hence nilpotent.
So, we apply induction hypothesis to V/W. We get a basis $\overline{B} = \{v_1 + W, \ldots, v_k + W\}$.
Note that $\overline{x}(v_j + W) = \alpha_1 v_+ \cdots + \alpha_{j-1} v_{j-1} + W$. Take the basis $B = \{w, v_1, \ldots, v_k\}$.
We get $x(v_j) = \alpha_0 w + \cdots + \alpha_{j-1} v_{j-1}$. And done! It satisfies the Upper triangular matrix! □

Theorem 3.2. Let V be a vector space. Suppose that L is a Lie subalgebra of $\mathrm{gl}(V)$ such that every element of L is a nilpotent linear transformation of V . Then there is a basis of V in which every element of L is represented by a strictly upper triangular matrix.

Proof. But before proving this, we prove the following lemma.

Lemma 3.3. Suppose that $L \subset \mathrm{gl}(V)$ is such that every $x \in L$ is nilpotent. Then $\exists 0 \neq v \in V$ such that $x(v) = 0 \forall x \in L$.

Proof. We do induction on L. Say dim L is 1. Then by the above we showed that there is some non zero $v \in V$ such that $z(v) = 0$. But L is spanned by z. Then done.
Say it is true for all lie algebra of dimension upto k. Suppose dim $L = k + 1$.

Claim 3.4. There is an ideal $I \subset L$ such that dimension of I is k.

Proof. Let A be maximal lie subalgebra of L. Consider L/A.
Consider the linear map $\overline{ad} : A \to \mathrm{gl}(L/A)$ such that

$$\overline{ad}_a(x) = [a, x] + A.$$

Claim 3.5. $\overline{ad}_a$ is a lie homorphism.

Proof. To show it is a homorphism, note that

$$\overline{ad}(\alpha a + \beta b) = \alpha\overline{ad}(a) + \beta\overline{ad}(b).$$

Note that

$$\begin{aligned}\overline{ad}(\alpha a + \beta b)(x) &= [\alpha a + \beta b, x] + A = \alpha[a, x] + \beta[b, x] + A \\ &= \alpha\overline{ad}(a) + \beta\overline{ad}(b).\end{aligned}$$

To show it is a lie homorphism, we have to show

$$\overline{ad}([a, b]) = [\overline{ad}(a), \overline{ad}(b)].$$

Or show

$$\overline{ad}([a, b])(x) = [\overline{ad}(a), \overline{ad}(b)](x)$$

But LHS is $[[a, b], x] + A$ and RHS is

$$(\overline{ad}(a) \cdot \overline{ad}(b) - \overline{ad}(b) \cdot \overline{ad}(a))(x)$$

$$= [a, [b, x]] - [b, [a, x]] + A = [a, [b, x]] + [b, [x, a]] + A = -[x, [a, b]] + A = [[a, b], x] + A.$$

□

So

$$\mathrm{img}(\overline{ad}) \subset \mathrm{gl}(L/A) \implies \dim\ \mathrm{img}(\overline{ad}) \leq dim(A) < dimL = k + 1.$$

So dim $\mathrm{img}(\overline{ad}) \leq k$ and $a \in A$ is nilpotent $\implies ad(a)$ is nilpotent $\implies \overline{ad}(a)$ is nilpotent.
So $\mathrm{img}(\overline{ad})$ satisfies induction hypothesis. Hence there exists $y + A \neq 0 \in L/A$ such that

$$(\overline{ad})_a(y + A) = 0 \forall a \in A \implies [a, y] + A = 0 \implies [a, y] \in A \forall a \in A.$$

Note if $A \subset A \bigoplus Fy \subset L$ then since A has maximum dimension we get that $A \bigoplus Fy = L \implies dimA = dimL - 1$ and note that A is ideal here. □

So $L = A \bigoplus Fy$. So $dim(A) = k$. We apply induction hypothesis on A. So there exist $0 \neq u \in V$ such that $a(u) = 0 \forall a \in A$.
Let $W = \cap_{a \in A} \text{Null}(A)$. Note $u \in W$.
By invariance lemma, W is invariant under L. So $y(W) \subset W$. But y is nilpotent. So is y restricted over W is nilpotent.Hence $\exists v \in W$ such that $y(v) = 0$. Since

$$L = A \bigoplus Fy \implies x = a + By, a \in A, B \in F \implies x(v) = a(v) + By(v) = 0 \forall x \in L.$$

□

We use induction on L. If dim $L = 1$. Then L is spanned by a vector x. Then x is representable as a Upper triangular matrix. So done.
Suppose $\forall$ lie algebra of dim $\leq k$, the statement is true. Say dim $L = k + 1$.
We showed that $\exists 0 \neq u \in V$ such that $x(u) = 0 \forall x \in L$. Set $U = span\{u\}$. And let $\overline{V} = V/U$. Consider the map $L \to gl(\overline{V})$ such that $x \in L \to \overline{x}$.

Note that the image of the map is subset of gl$(\overline{V})$. Moreover $\overline{V}$ has dimension k. So $\exists$ basis of $\overline{V}$ such that all $\overline{x}$ are upper triangular. Hence $\{v_1 + U, \ldots, v_k + U\}$. Then $\{u, v_1, \ldots, v_k\}$ is a basis of V. As $x(u) = 0$, we get that x is strictly upper triangular.

□

§4 second version of Engel's theorem

Theorem 4.1. A Lie algebra L is nilpotent if and only if for all $x \in L$ the linear map $adx : L \to$ is nilpotent.

Proof. If L is nilpotent then $\exists N \in \mathbb{N}$ such that $L^N = 0$. Then $[[[\ldots[x[x, \ldots [x, y] \ldots] \in L^N = 0 \implies (adx)^{N-1}(y) = 0 \implies (adx)^{N-1} = 0$.
Let $\overline{L} = ad\ L$. $ad : L \to \text{gl}(L)$. Every element of $\overline{L}$ is nilpotent. So $\exists$ basis such that $ad\ x$ is upper triangular strictly.
So $\overline{L}$ is nilpotent. (as when we commute two Upper triangular matrix, we get 2nd upper diagonal to be 0s and so on). Since $\overline{L}$ is nilpotent, then so is $L/Z(L) \cong \overline{L}$. Hence L is nilpotent. □

Remark 4.2. Converse of Engel's theorem is not true. Let I denote the identity map in gl(V). The Lie subalgebra Span $\{I\}$ of gl(V) is nilpotent.It is (trivially) nilpotent as it is spanned by one vector and $[x, x] = 0$. In any basis of V , the map I is represented by the identity matrix, which is certainly not strictly upper triangular.

§5 Lie's theorem

Theorem 5.1. Let V be an n-dimensional complex vector space and let L be a solvable Lie subalgebra of gl(V). Then there is a basis of V in which every element of L is represented by an upper triangular matrix.

Claim 5.2. Suppose $V \equiv \mathbb{C}^n$ and $x \in \text{gl}(V)$. Then $\exists$ a basis of V such that x is upper triangular matrix.

Proof. We begin with the claim

Claim 5.3. x has an eigenvector

Proof. Take any $0 \neq v \in V$ and consider$\{v, xv, x^2v, \ldots, x^m v\}$ where m is the minimum such that the vectors are linearly dependent. So $\exists \alpha_0, ;\alpha_m$ such that $\alpha_m \neq 0$. We can factor over $\mathbb{C}$. So

$$\alpha_m(x - \lambda_0 I) \ldots (x - \lambda_m I)v = 0.$$

Take k to be minimum such that $w = (x - \lambda_{k+1} I) \ldots (x - \lambda_m I)v = 0$. Now, $(x - \lambda_k I)w = 0 \implies xw = \lambda_k w$. ☐

Now we try to prove by induction on dimension of V. Say it is true for all dimensions less than or equal to k. Let $w \in V$ be an eigenvector of x with value λ.
Consider $x : V \to V$ and $\overline{x} : V/\mathbb{C}w \to V/\mathbb{C}w$. And $\overline{x}(v + \mathbb{C}w) = x(v) + \mathbb{C}w$. So $\dim(V/\mathbb{C}w) = k$. We apply induction hypothesis to $V/\mathbb{C}w$. Then there is a basis $\{v_1 + \mathbb{C}w, \ldots, v_{k+1} + \mathbb{C}w\}$. Then the basis of V as $\{w, v_1, \ldots, v_{k+1}\}$ works. ☐

Lemma 5.4. Let V be a non-zero complex vector space. Suppose that L is a solvable Lie subalgebra of $gl(V)$. Then there is some non-zero $v \in V$ which is a simultaneous eigenvector for all $x \in L$.

Proof. We do Induction on dim L. If 1 then say it is spanned by x. Since $x \in \mathrm{gl}(v)$ it has an eigenvector. So done.
Suppose the statement holds for all lie algebra of dim k and dim $L = k + 1$.
Since L is solvable $\implies L^{(N)} = 0$ for some $N \in \mathbb{N}$. Note $L' \subset L$ and $L' \neq L$ else $L^{(N)} = L \forall n$.
Choose a subspace A of L which contains L' and is such that $L = A \bigoplus Span\{z\}$ for some $z \in L$.

Claim 5.5. A is ideal of L.

Proof.

$$x \in L, a \in A, [x, a] \in [L, L] = L' \subset A'.$$

☐

dim $A = k$, A is solvable. By inductive hypothesis $\implies \exists w \in V$ w is eigenvector for all $a \in A$.

$$\lambda : A \to \mathbb{C}$$

$$aw = \lambda(a)w$$

$$V_\lambda = \{v \in V | a(v) = \lambda(a)v\}$$

So by invaraince lemma, we get that V_λ is Linvariant. So $x(v) \in V_\lambda \forall v \in V_\lambda$. So $\exists u \in V_\lambda$ which is eigenvector of $z \in V$. Let $z(u) = \mu(u)$. $\forall x \in L, x = \alpha + \beta z$

$$x(u) = \alpha(u) + \beta z(u) = \lambda(\alpha)u + \beta\mu(u)\forall x.$$

So done! ☐

So now the main proof.

Proof. We do induction V. For dim 1 it is good. Say true for k. Now for $k+1$. It is essentially same as engel's theorem.
Find $w \in V$ such that w is eigenvector for all $x \in L$.

$$\implies x(w) = \lambda(x)w, \lambda : V \to \mathbb{C}.$$

Define

$$\overline{x}(V + \mathbb{C}w) = x(v) + \mathbb{C}w.$$

Consider $Im(\overline{x}) \subset \mathrm{gl}(V/\mathbb{C}w)$. So we use induction hypothesis, get a basis $\{v_1 + \mathbb{C}w, \ldots, v_k + \mathbb{C}w\}$. So let the basis be $\{w, v_1, \ldots, v_k\}$. □