“NOT” INVARIANTS

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What are Invariants ?

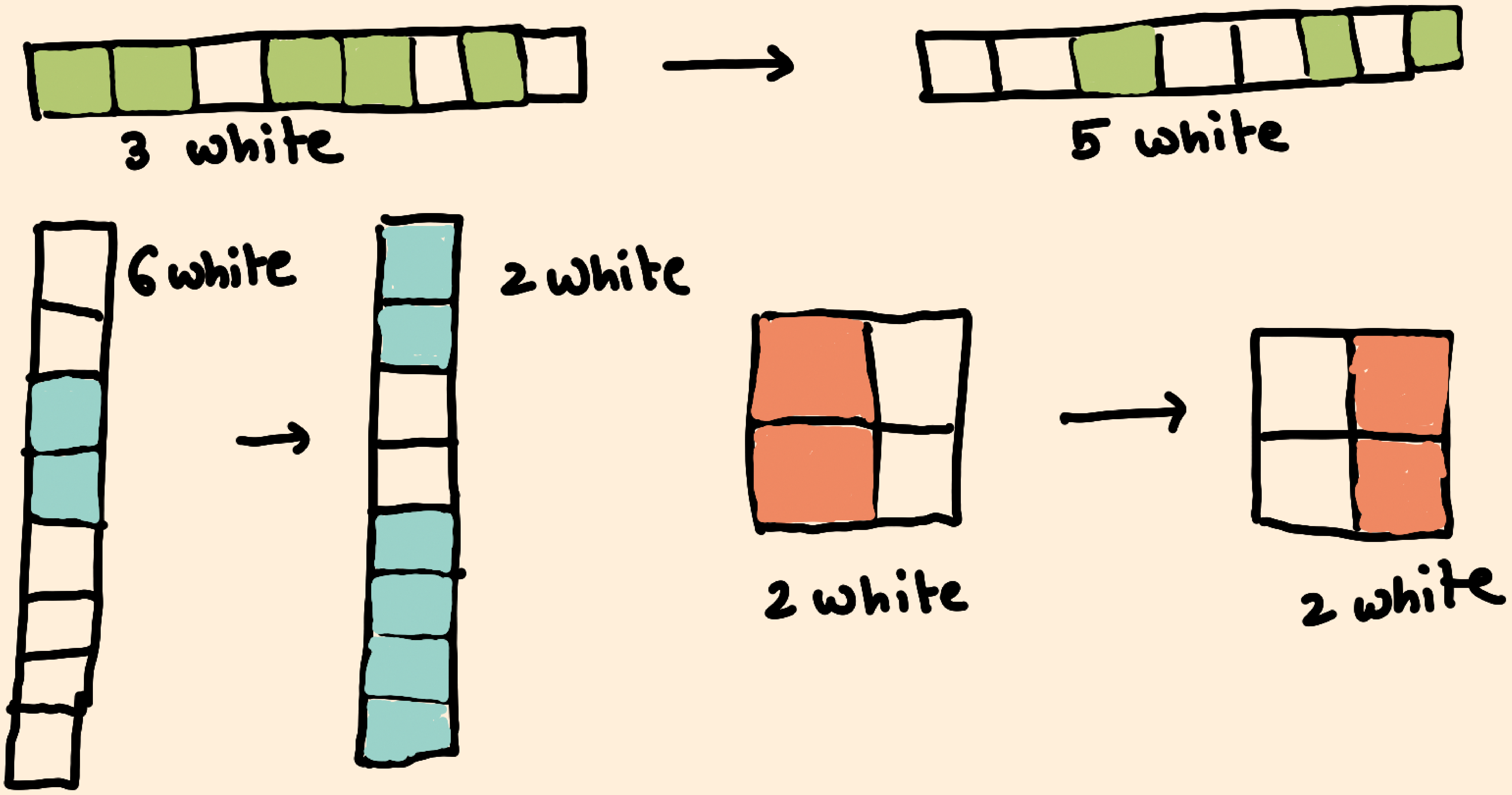
- During some kind of process , some object remains constant.
- Anything can be the "object": say ; parity of sum of the numbers , etc . . . .
- It is used a lot when we want to show using the process in not possible to achieve this particular outcome.

Example 1: Dominoes in Chess

An 8×8 chess board is coloured in the usual way, but that's boring. You can take any row, column or 2×2 square & reverse the color inside it, switching black to white & white to black.

Prove that it is impossible to end up with 63 white and 1 black square.

Working : Dominoes in Chess



Our invariant object would be :

parity of number of white squares in the board.

Solution : Dominoes in Chess

Note that any of these operations changes the total number of white squares by an even number.

In case of row & column operation, it goes from

$$k \longrightarrow 8-k$$

So the change is $8-2k = 2(4-k)$.

In case of 2×2, we go from

$$k \to 4-k$$

so the change is $4-2k = 2(2-k)$.

Since, we start with an even number of white squares, it is impossible to end with 63 (odd) white squares.

Example 2: Putnam 2008

Start with a sequence $a_1, a_2, \dots, a_n$ of positive integers. If possible, choose two indices $j<k$ such that $a_j \nmid a_k$ and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$. Prove that if this process is repeated, it must eventually stop.

Working / Ideas

What is invariant? We know that the product of all the numbers is invariant as

$$GCD(a,b) \cdot LCM(a,b) = a \cdot b$$

We can use this, or even better consider the primes dividing it.

Solution 2: Putnam 2008

let p be a prime such that $\exists\, i$ such $p \mid a_i$.

let $p^{e_1}, \ldots, p^{e_n}$ be the highest powers

of p that divide $a_1, \ldots, a_n$.

Then replacing a_j and a_k by their GCD

and LCM means replacing p^{e_j} and p^{e_k}

by $p^{\min\{e_j, e_k\}}$ and $p^{\max\{e_j, e_k\}}$.

So each step leaves the p-power sequence unchanged or more odered. But the latter can happen only finitely many times.

And there are only finitely many primes dividing some a_j.

So only finite amout of steps possible.

We will not be dealing with such invariants today.

Instead, we will deal with

knots

Woops!

I meant **knots**

(They are actually homeomorphic :))

But,

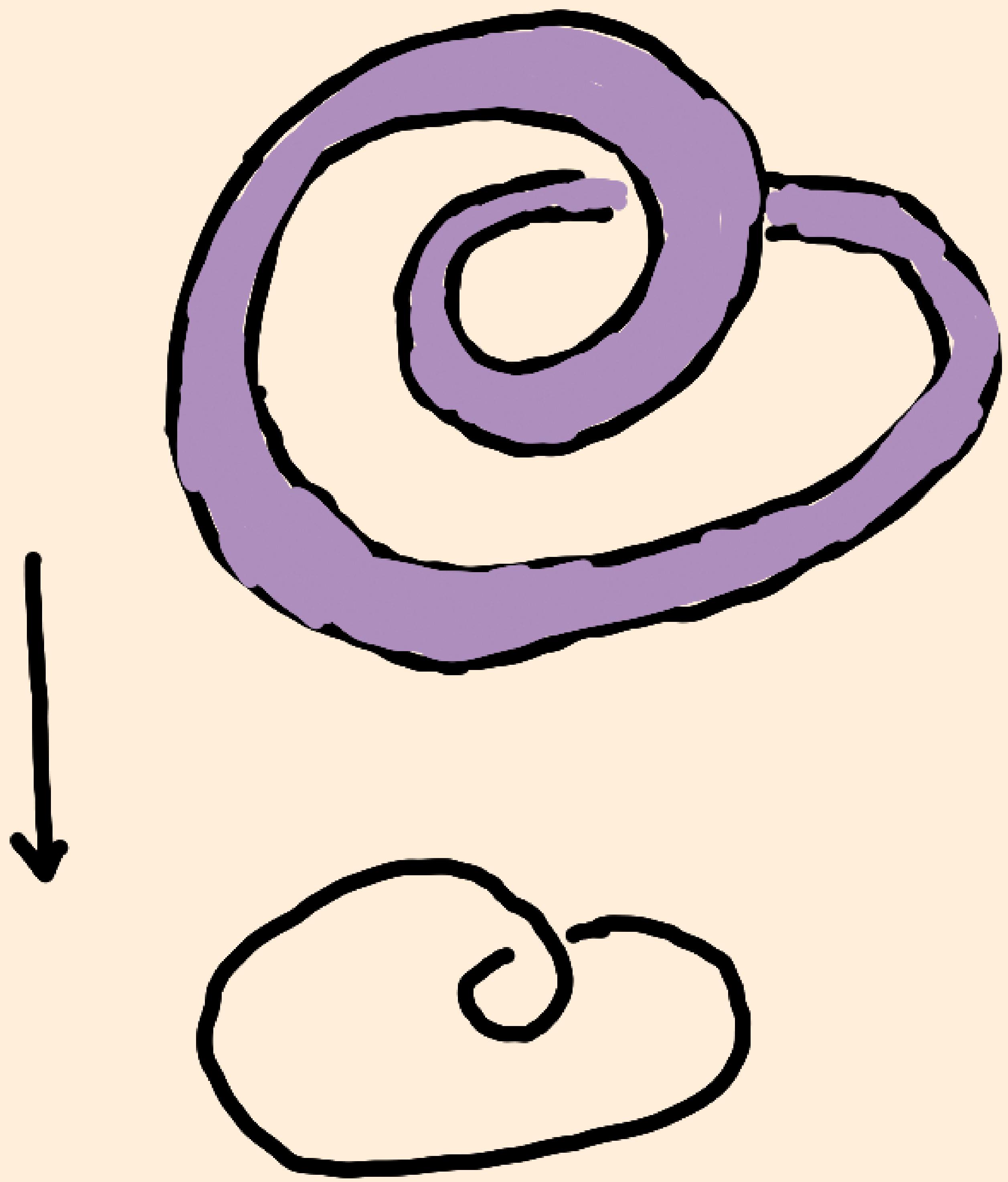
What are knots?

What are knots?

A knot is an embedding of the circle (S^1) into 3-D Euclidean space $\mathbb{R}^3$.

In simple terms, we can think knot as a knotted loop of string (which has no thickness).
It is a closed curve in space that does not intersect with itself.

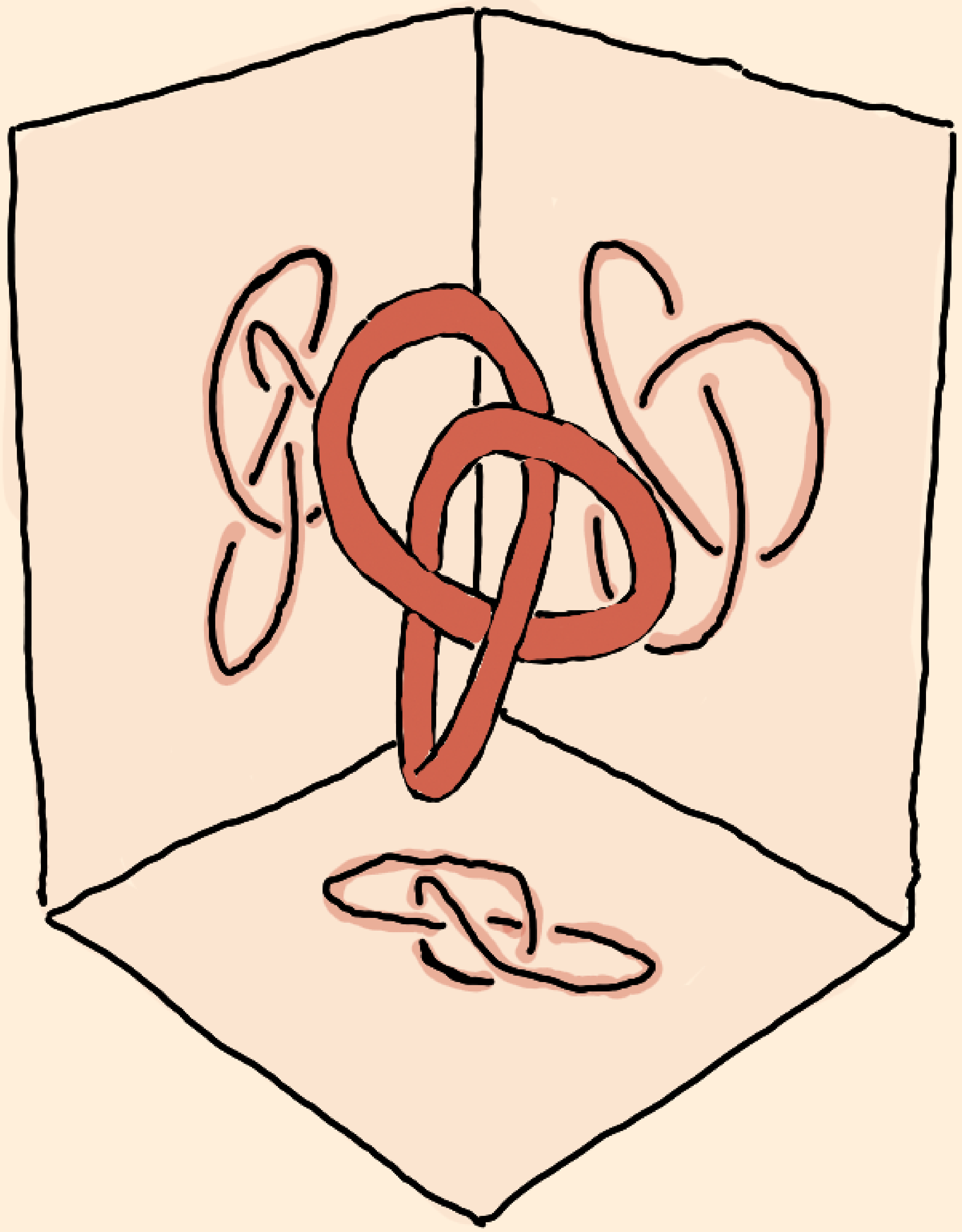
knot projections



A projection is a 2-dimensional picture of a 3-d knot.

Note: A knot can have large number of different projections.

An Example



What is Topology?

Topology is the area of mathematics concerned with the properties of a geometric object that are preserved under continuous deformations. Thus, we allow stretching, contracting, twisting, but not ruptures.

If object A can be transformed to object B using continuous deformations &

object B can be transformed to object A using continuous deformations then A & B are topologically indistinguishable.

Example : • circle & elipse wrt normal $\mathbb{R}$ - topology.

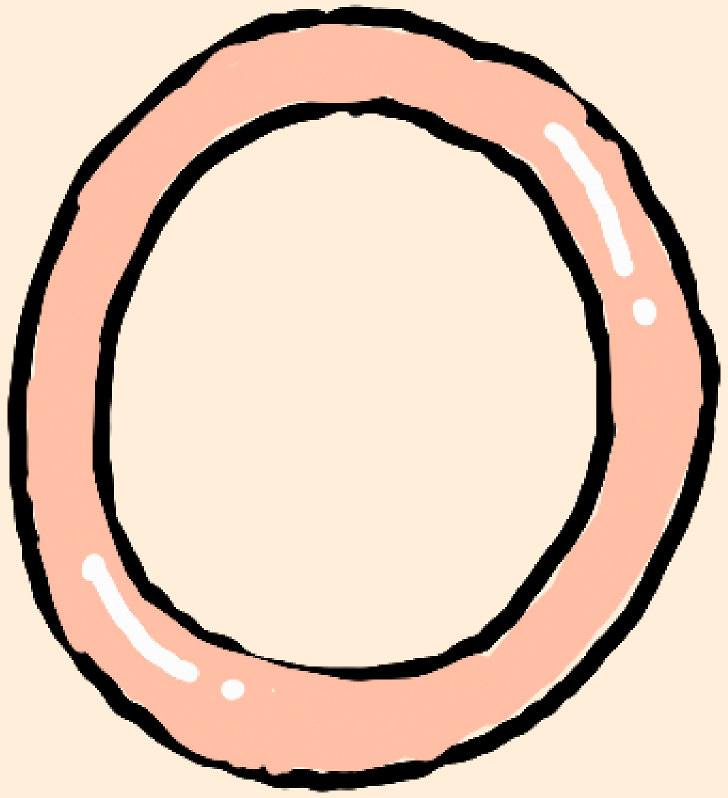
What is homeomorphism?

A function $f: X \to Y$, X & Y are topological spaces is homeomorphism if:

- f is a bijection
- f is continuous
- f^{-1} is continuous (So f is open map)

Note: These maps preserve all topological properties.

Examples of knots :

1. 

the unknot :

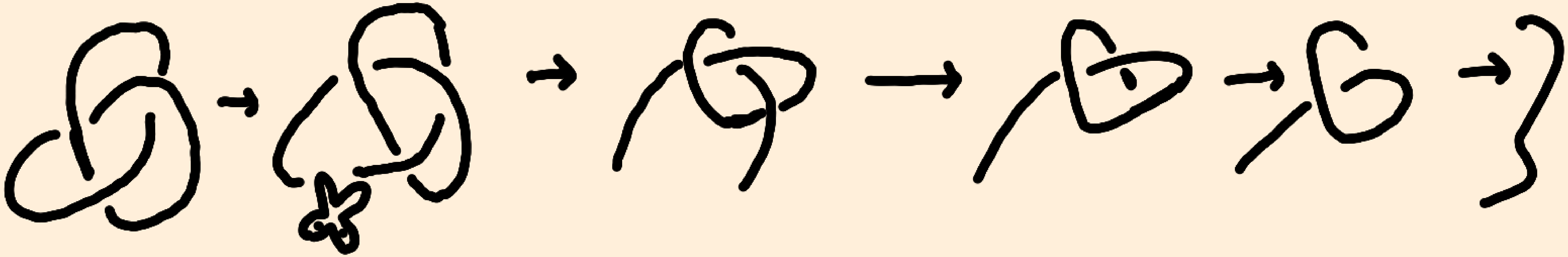
It is known as the trivial knot and is the simplest knot of all. It is simply a loop.

Connect two ends of a string together without crossing or knotting.

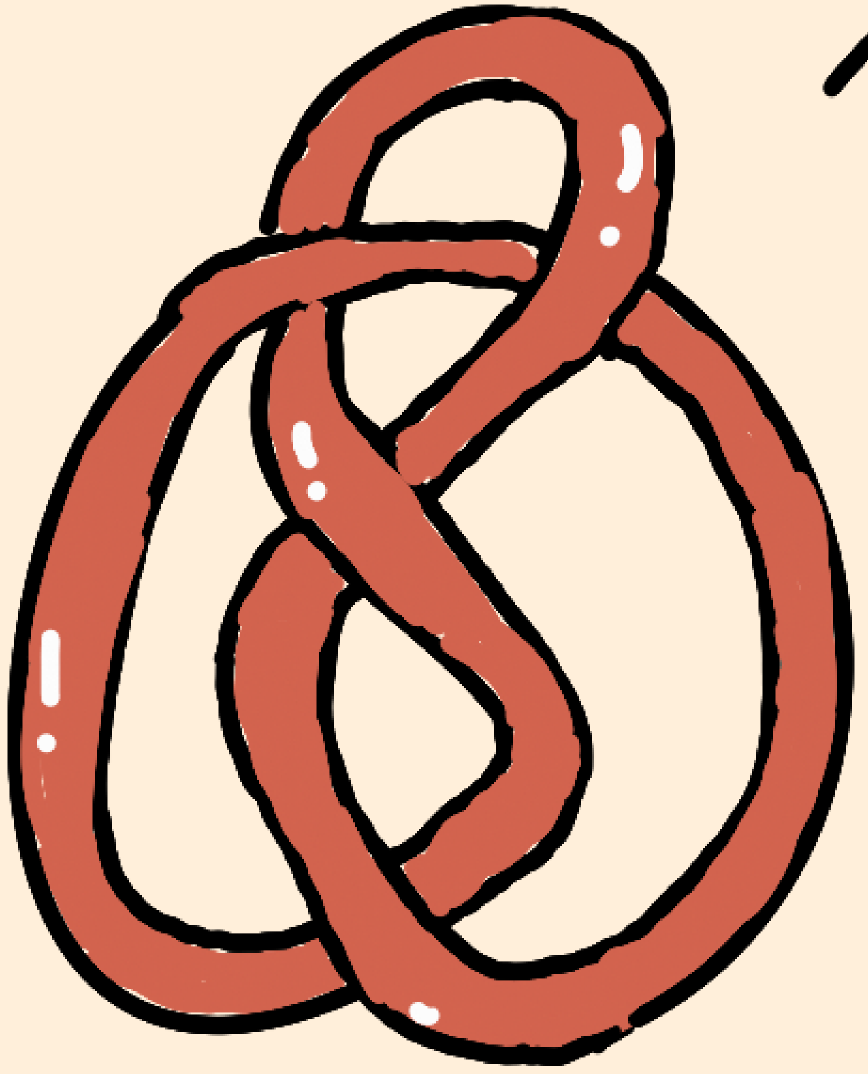
2. 

The trefoil:

Its simplest picture has 3 crossings. It is the closed end version of the simplest knot most people use to tie string.



3.



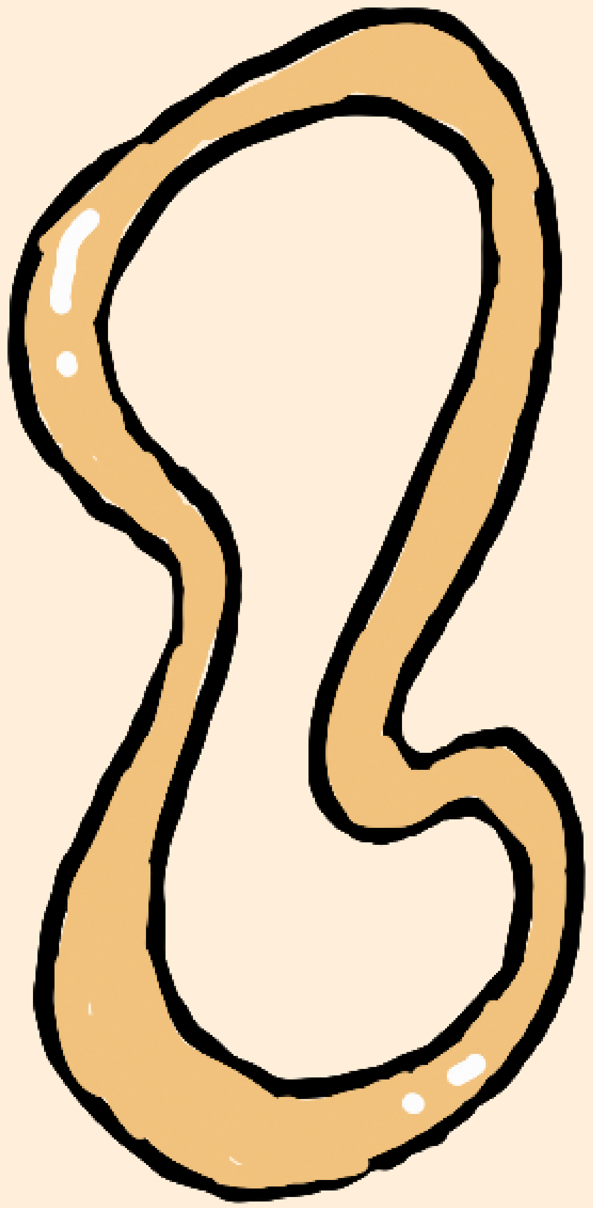
the figure-8 knot:

- Well, yeah....
- You guys try!

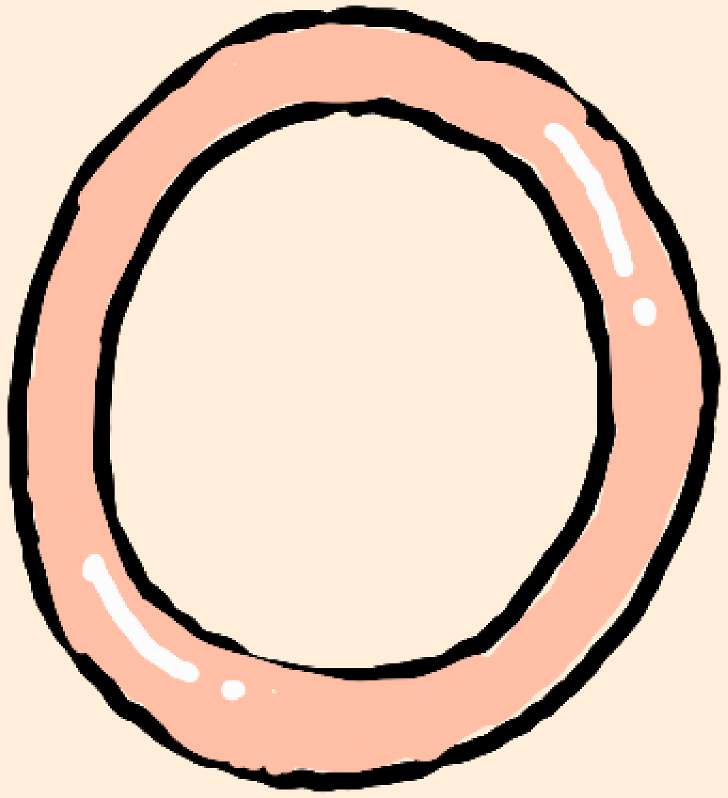
Equivalent knots

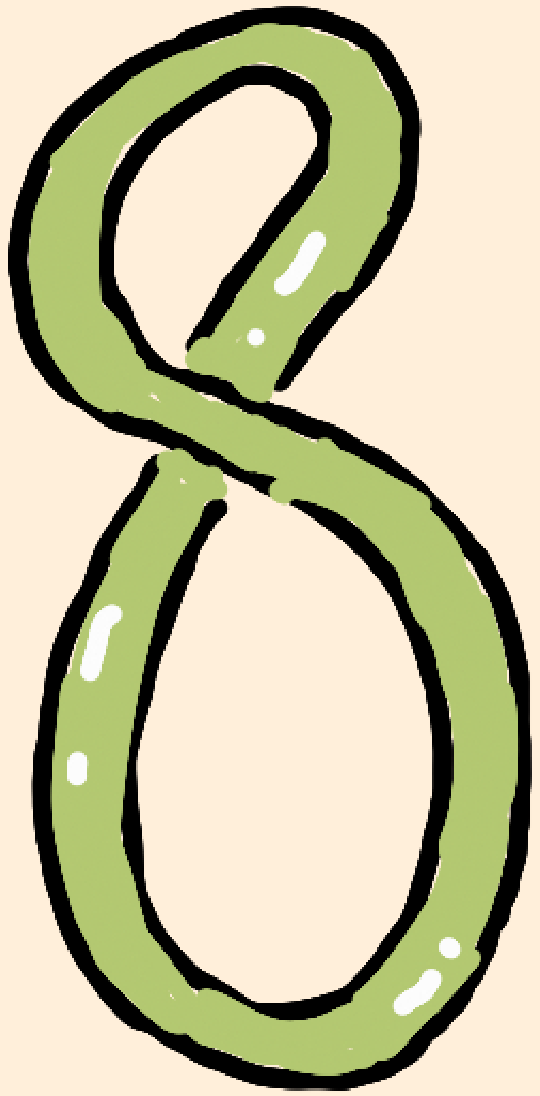
How would YOU like to define the conditions when two knots are equivalent?

Let us take Example

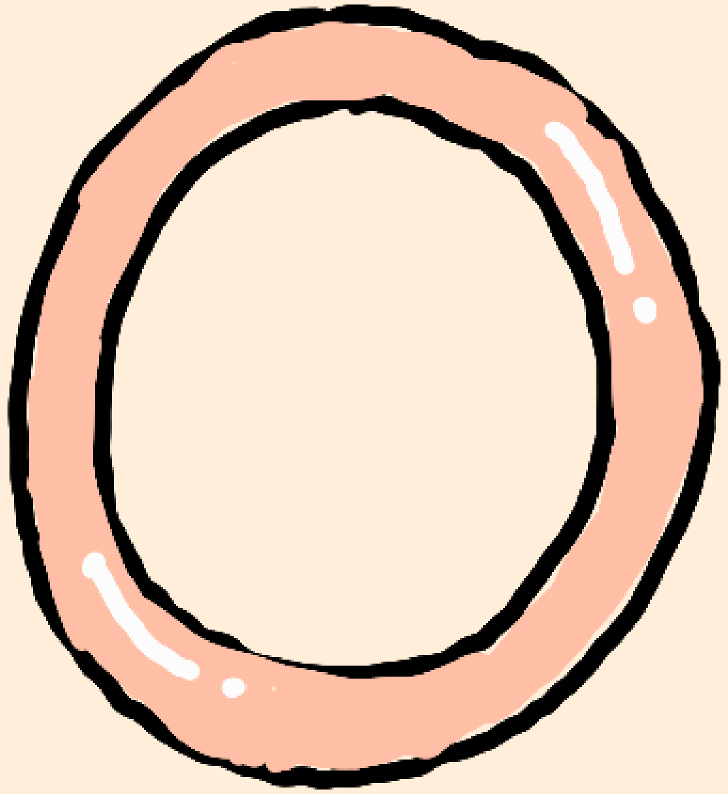


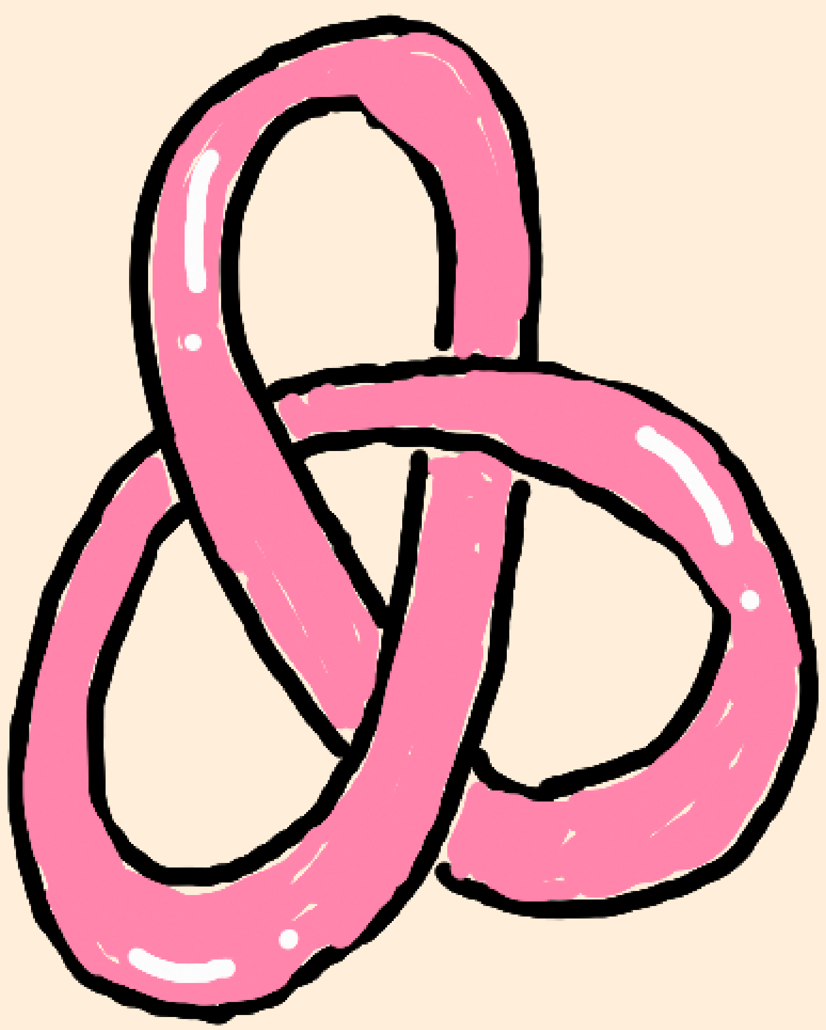
do you think these two are equivalent ?



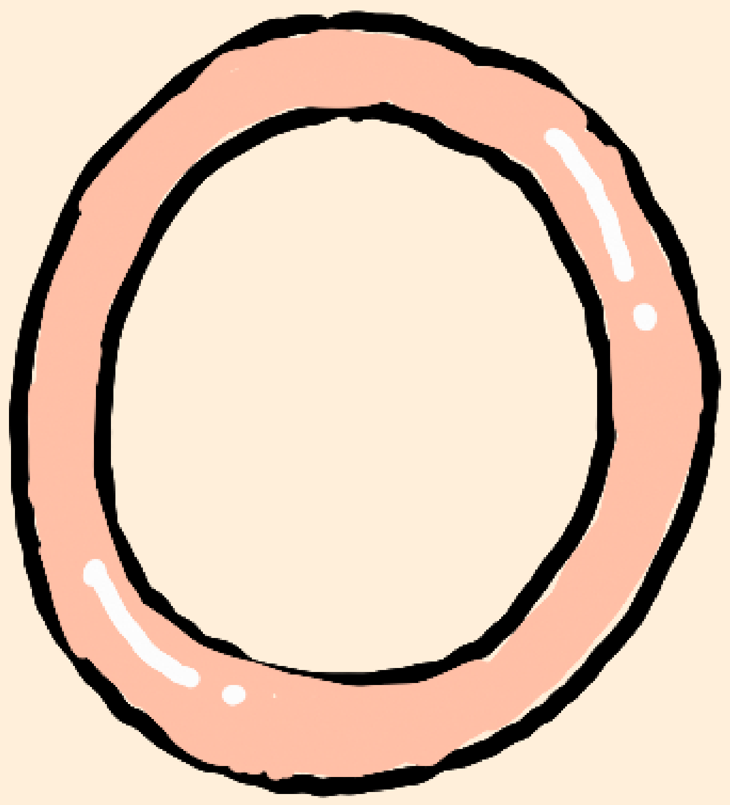


What about these two?





And, what about these two?



Equivalent knots

Two knots are considered to be the same, or, equivalent to one another, if you can 'deform' one into another without breaking the knots open.

Questions

1. How do we prove that two knots are the same?

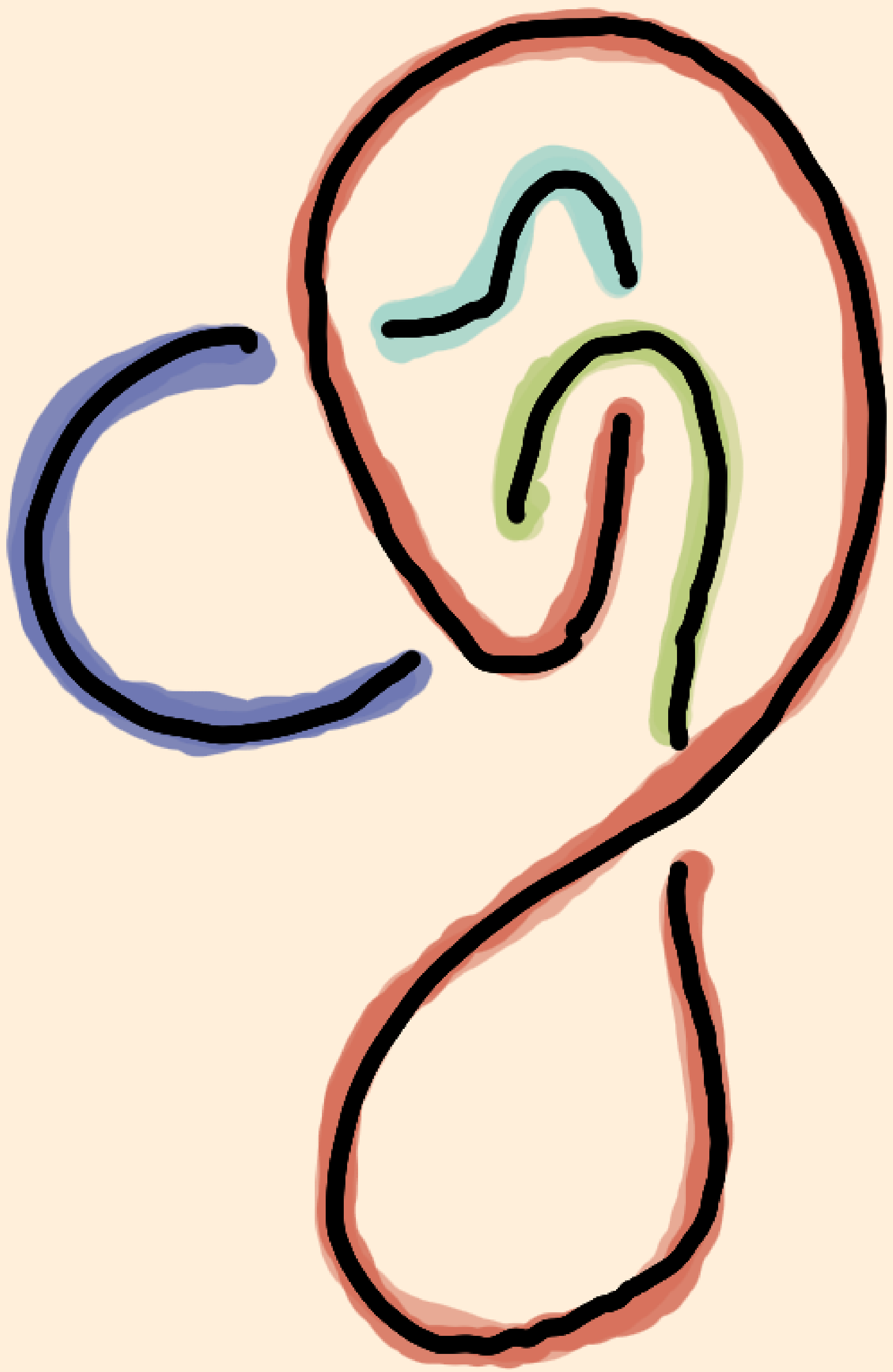
2. How do we prove that two knots are different?

Spoiler : Answering 2 is much harder than 1.

A few terminologies

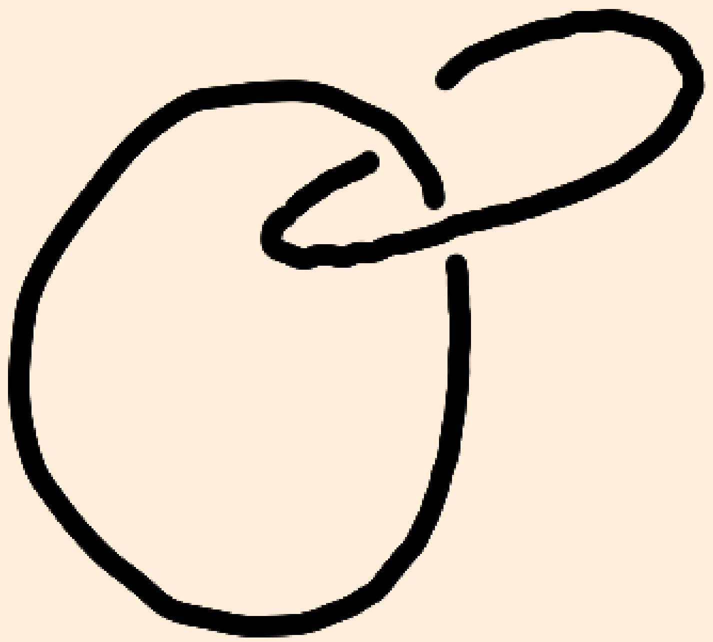
crossing: The place where two pieces of the Knot meet.

strand: A projection is a piece of the knot cuttoff on both ends by a crossing



All of the different coloured pieces of the knot represent strands

A link is a collection (or a union) of multiple knots.



→ 2 link

A planar isotopy of a projection is a manipulation of a part of the knot in 2D space (by shrinking, straightening, enlargin]) that does not change the number of crossings

An ambient isotopy of a projection is a manipulation of the knot.

But it should not be cut anywhere.

The Reidemeister Theorem

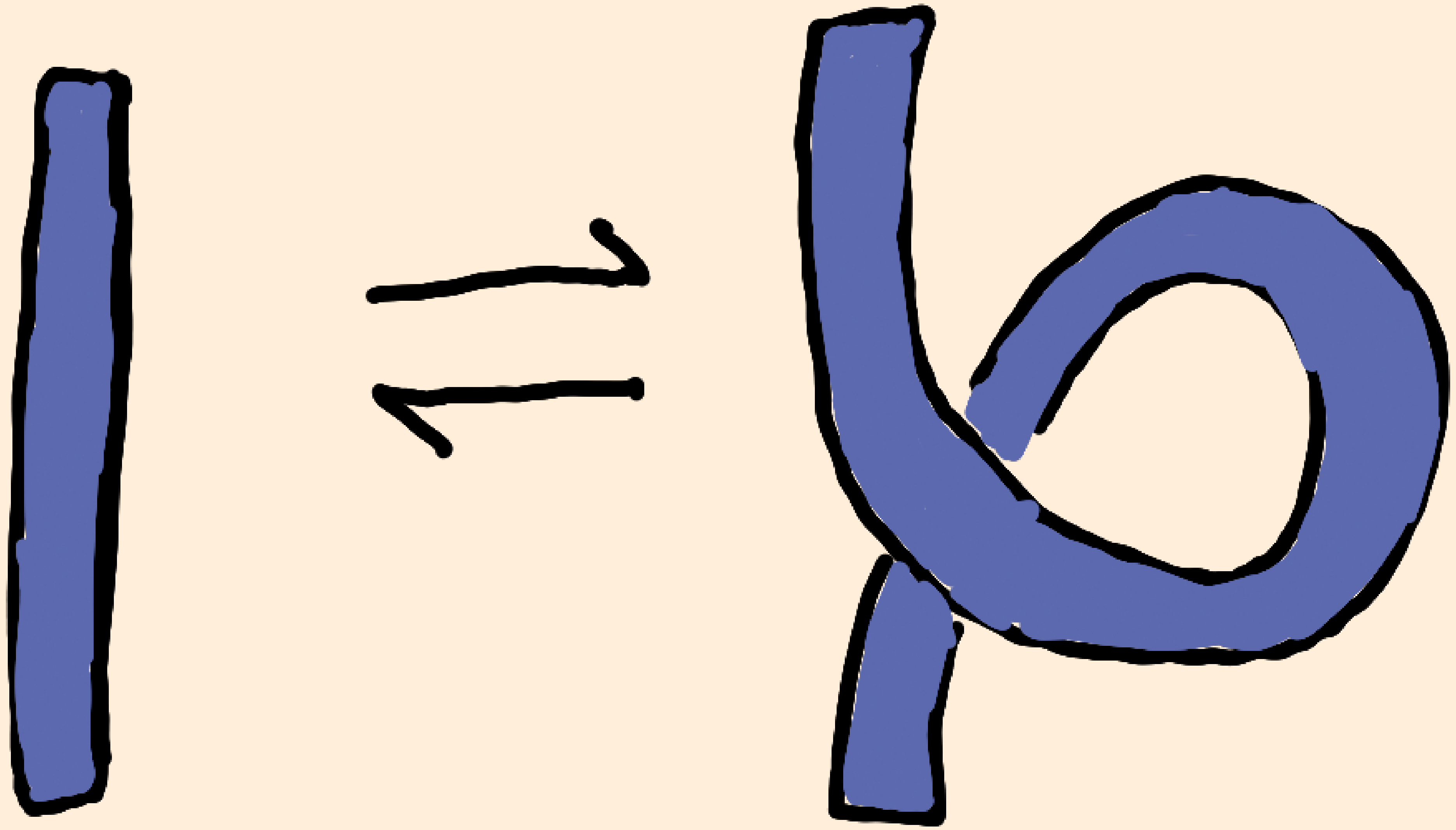
It states that two projections are of the same knot if and only if either of the projections can be transformed into the other using a series of planar isotopies and Reidemeister moves.

Turns out that there are <u>three</u> types of moves that encompass the ways we can manupilate a knot.

provided you do not cut

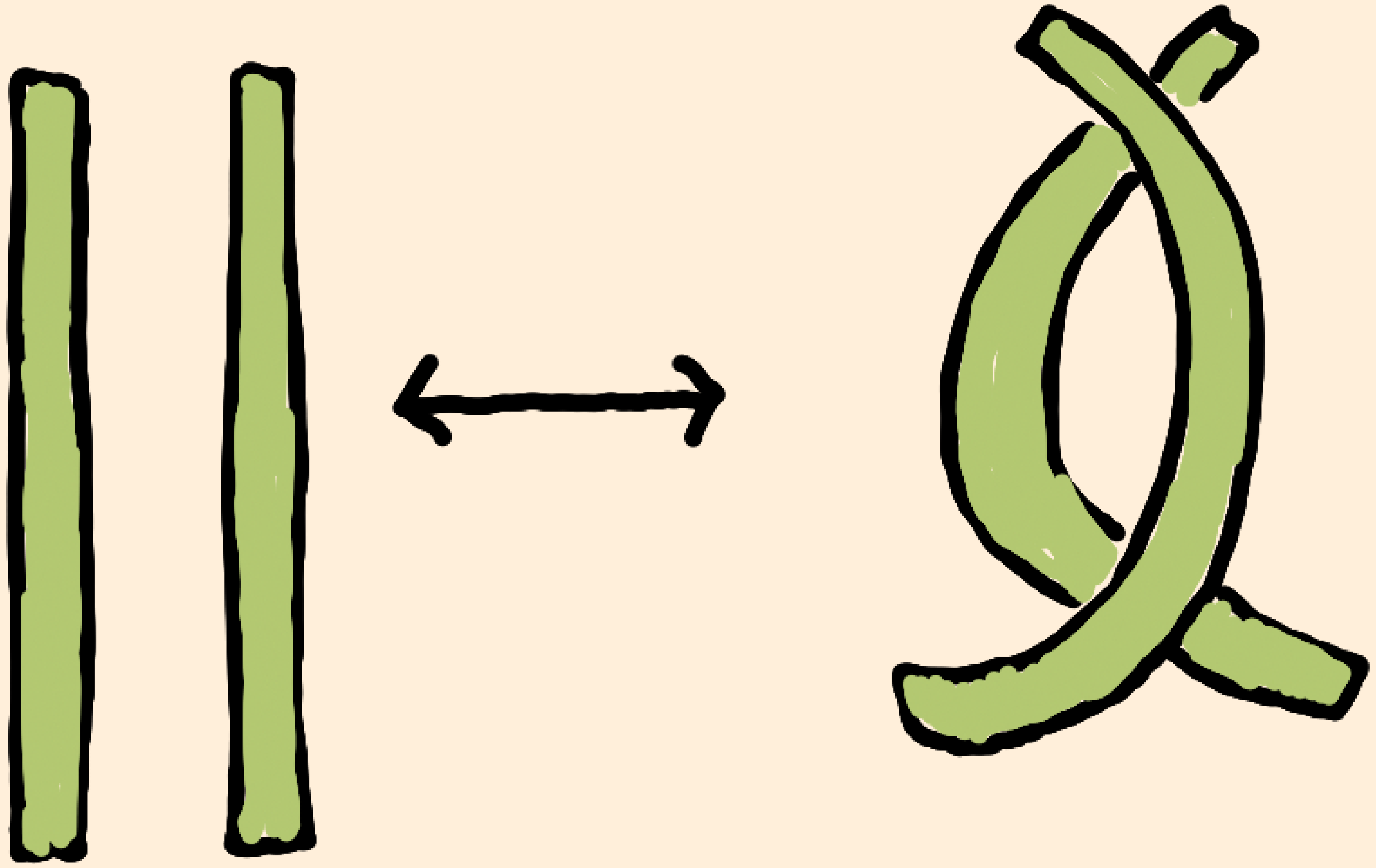
i. The Twist (RM I)

If you have a straight piece of the knot, and twist it once so as to create a single crossing.

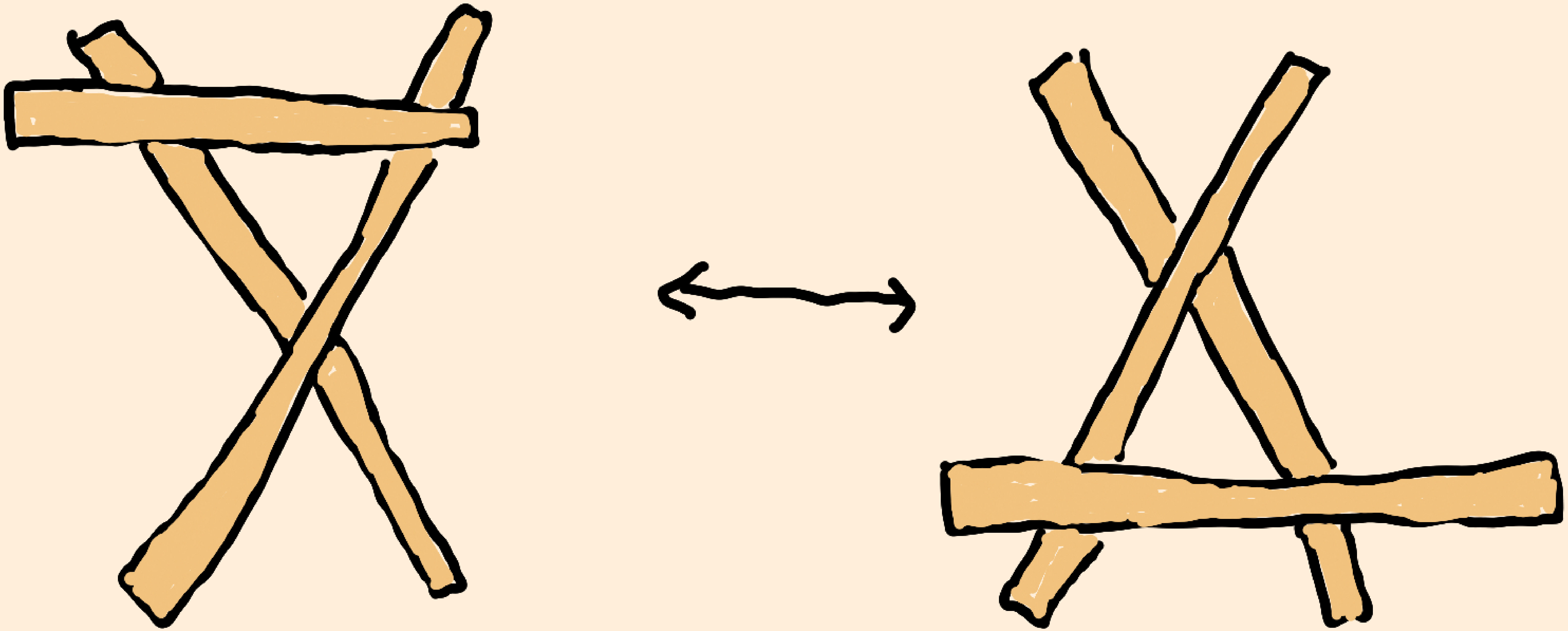


ii The Poke (RM 2)

We push one part of the knot under (or over) another part of the knot.



(iii) The slide (RM3)



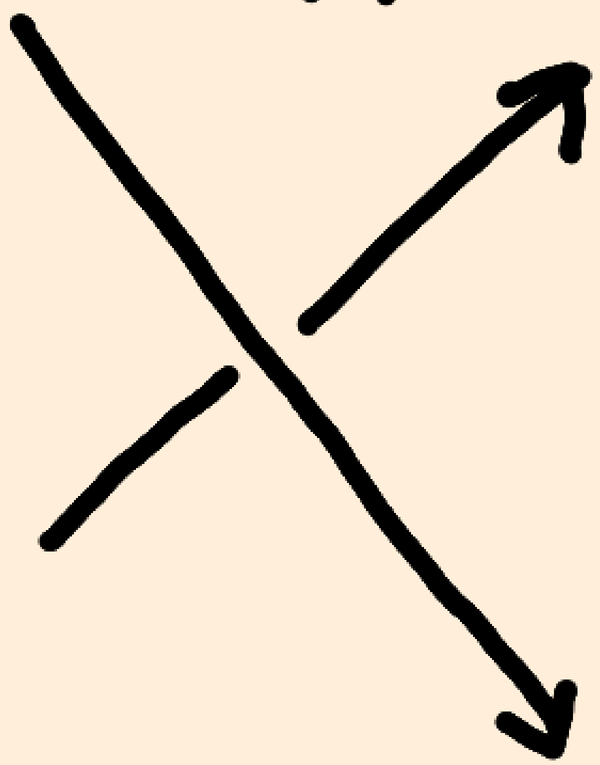
Linking Numbers

Now, we come to the first invariant!

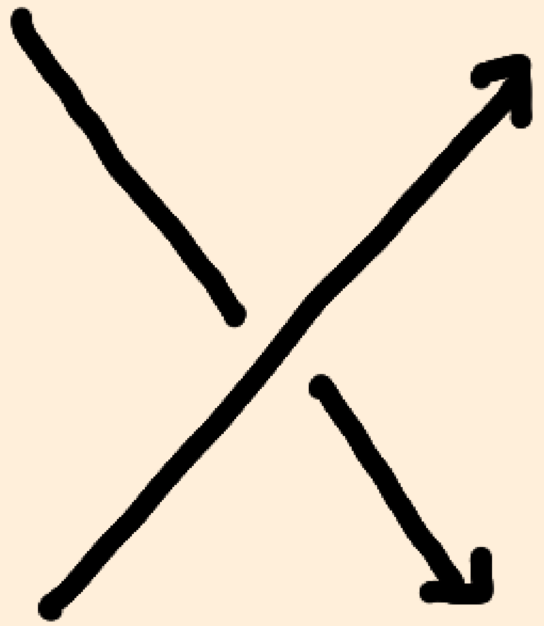
The linking number: We give $+1$ & -1 to crossing involving both knots and add them up (we need oriented link).

To orient a link, add a direction to string.

+1

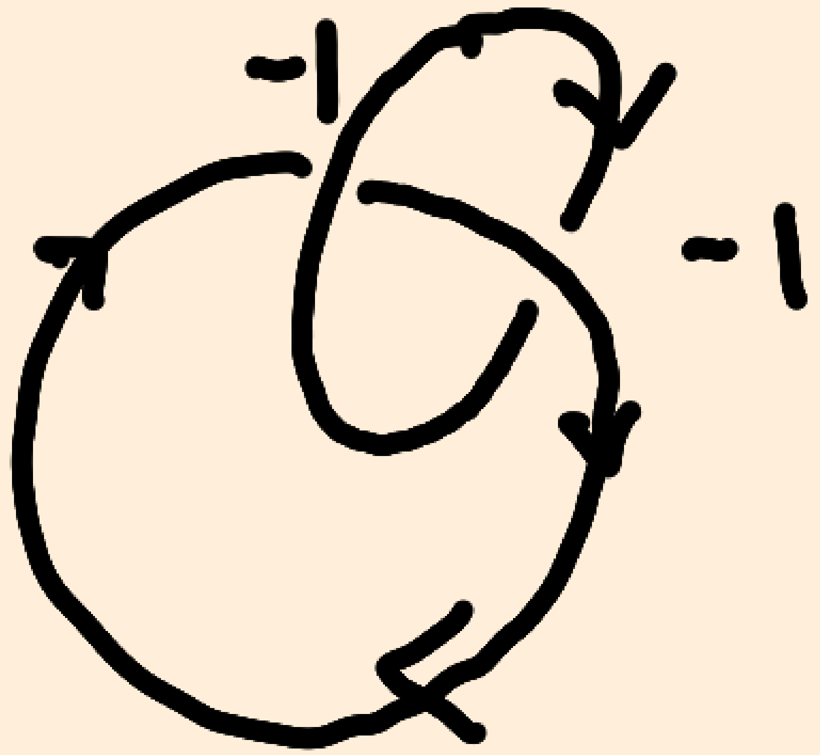


-1



Other configure is simply rotation of above two.

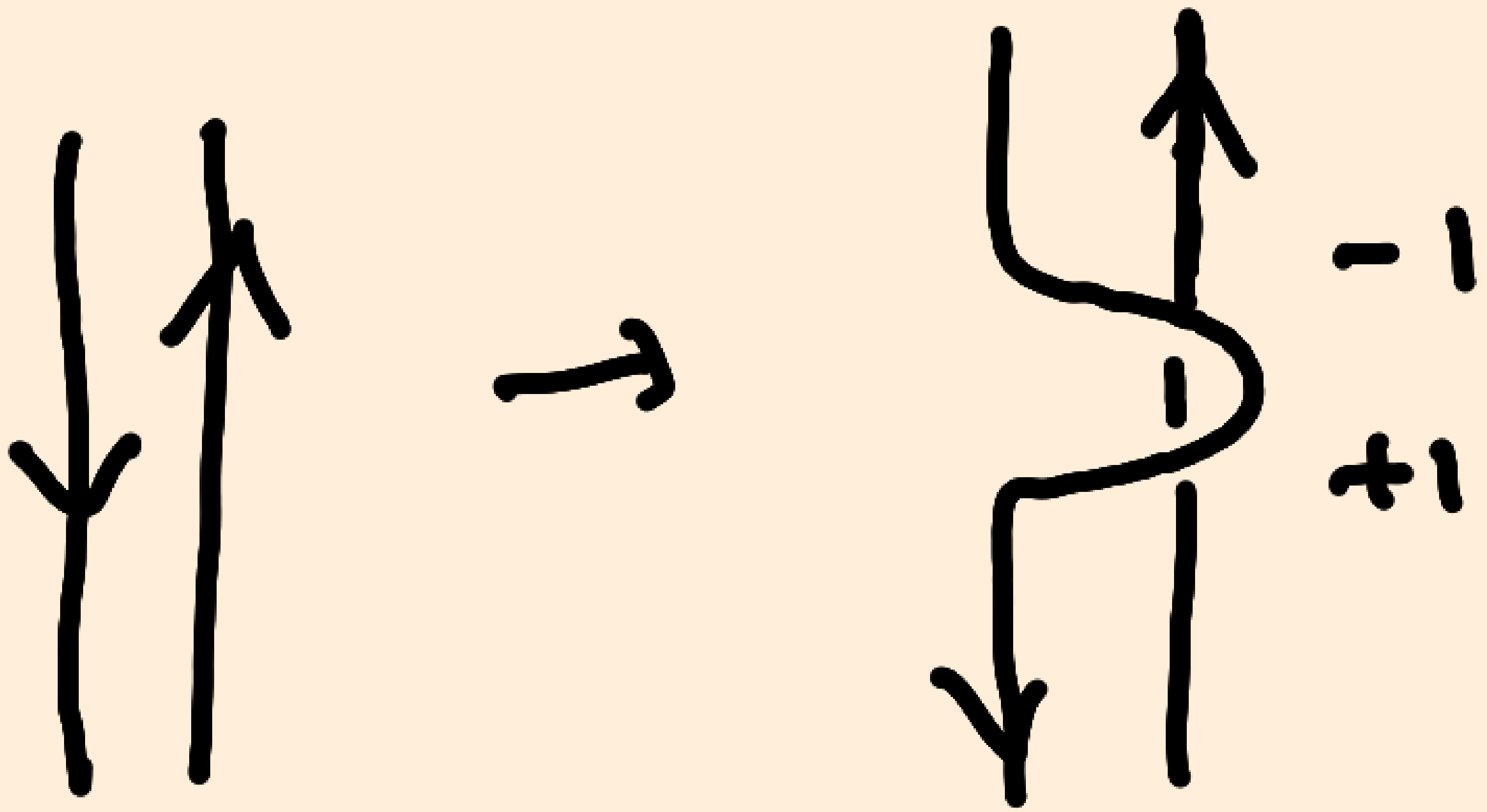
Example:



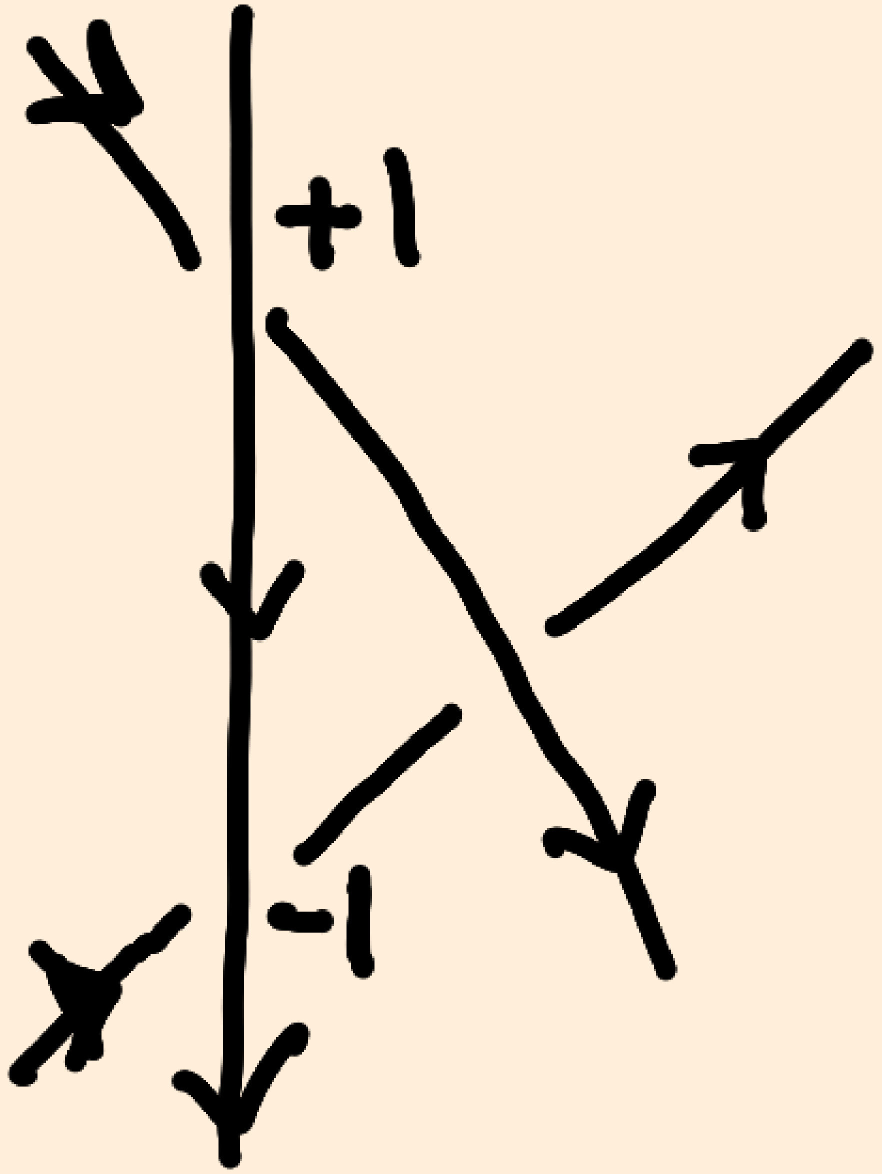
Theorem: Linking number is an invariant of knot diagram

proof: Reidemeister I only involve one knot.

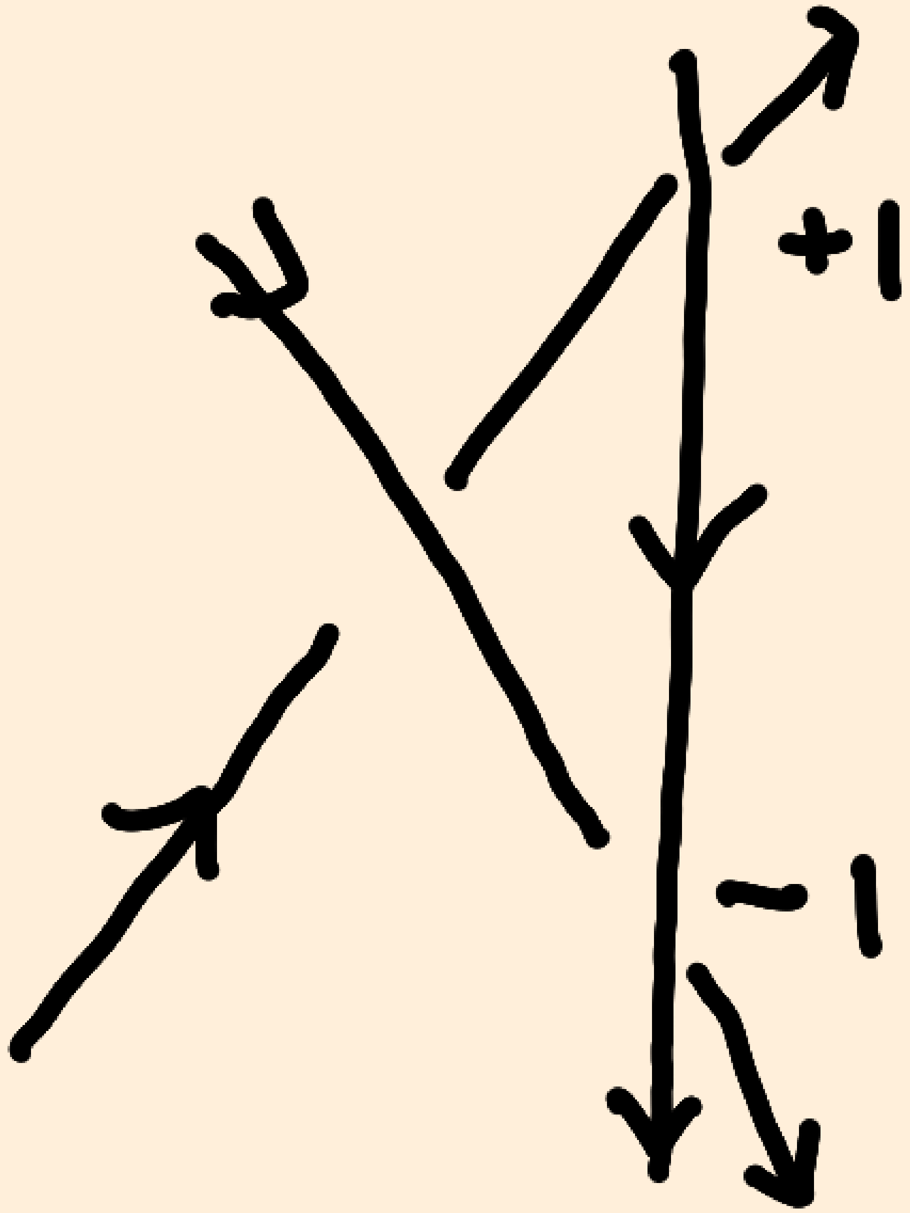
Note that for Reidemeister II, the crossings created and destroyed are of the opposite kinds. So they cancel out!



Redeimester III, the crossings created have the same net as the one getting deleted.



+1
-1



+1
-1

Tricolorability

So, we can do it for linked knots.

What about an invariant that distinguishes knot?

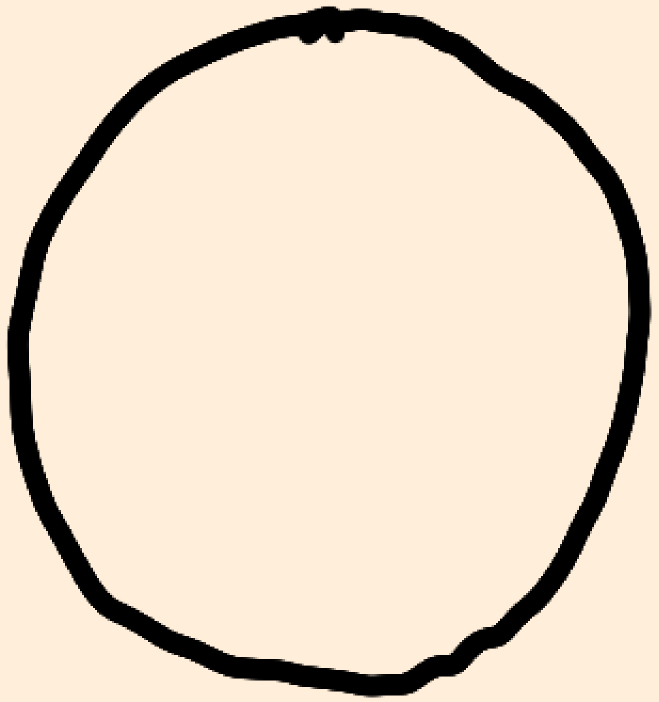
We have 3 colours.

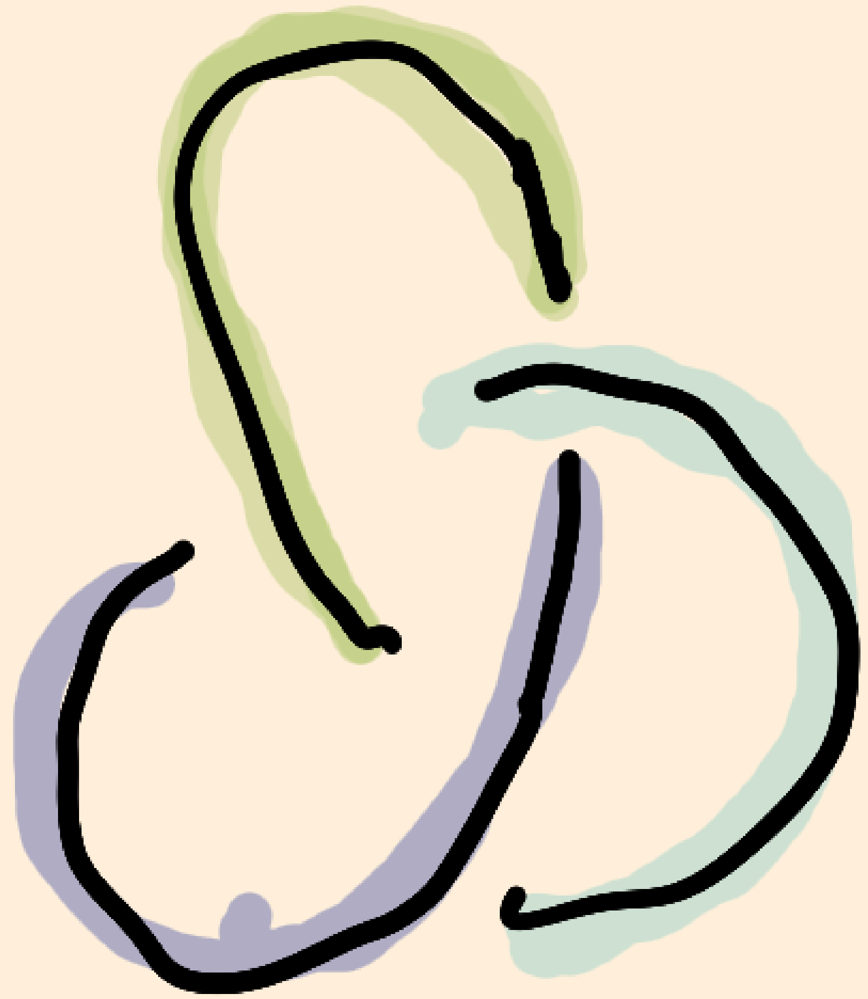
Definition: A knot diagram is tricolourable if each arc can be coloured with one of three colours such that:

(i) at least two colours are used in the diagram

(ii) at every crossing: all three colours are used or only one.

So an unknot is not tricolourable and trefoil is tricolourable.



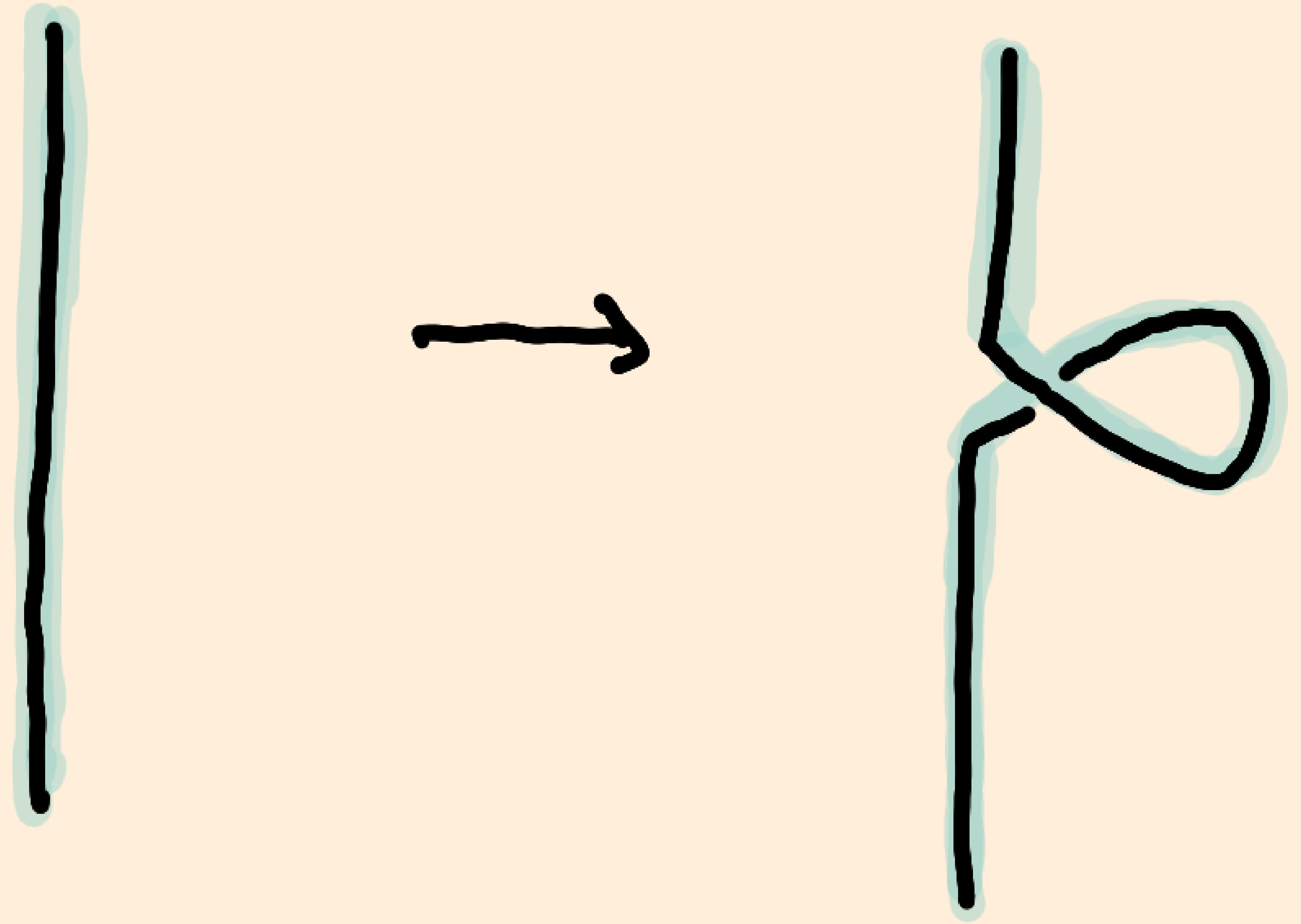


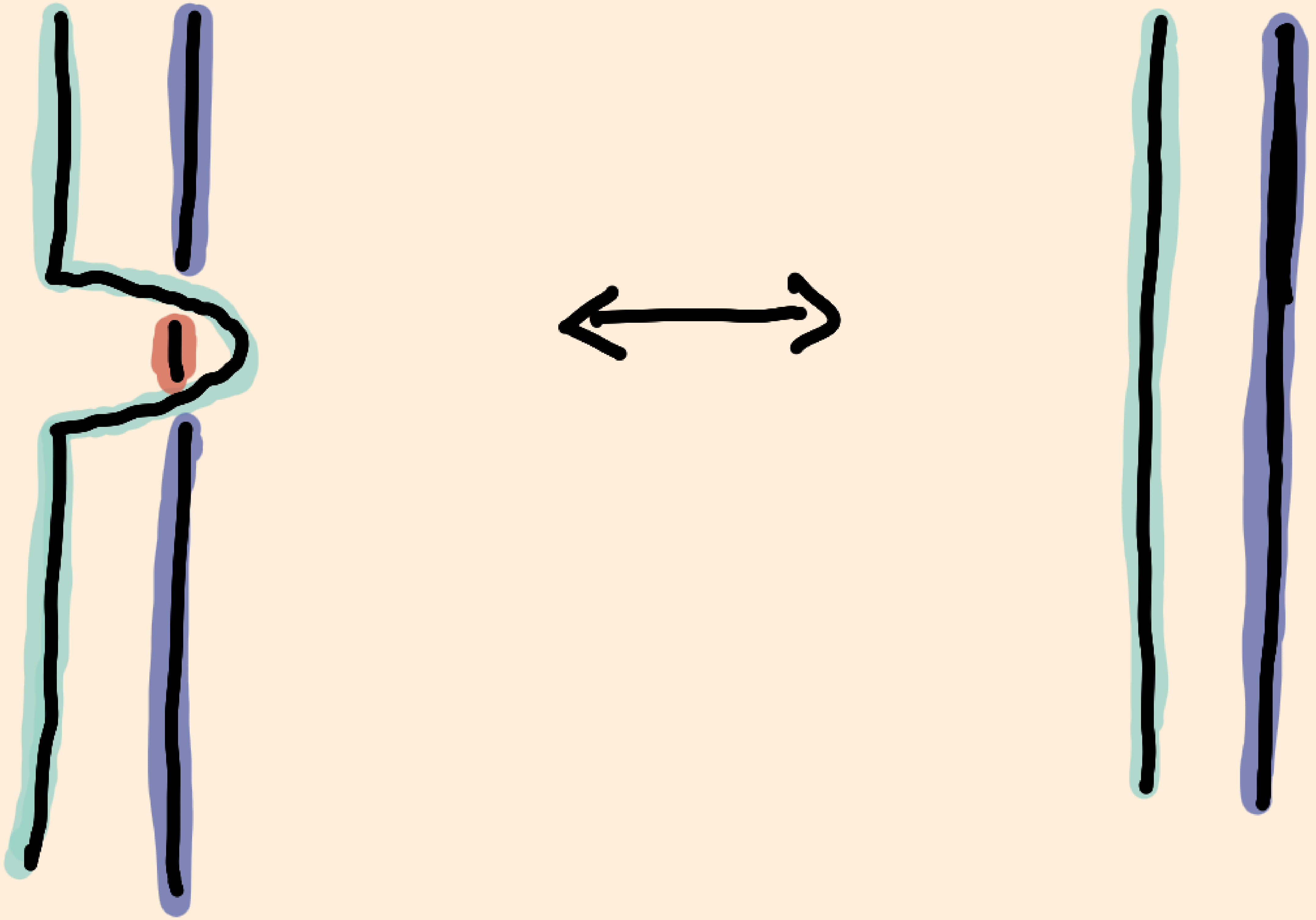
Theorem: Tricolourability is an invariant of knot diagrams

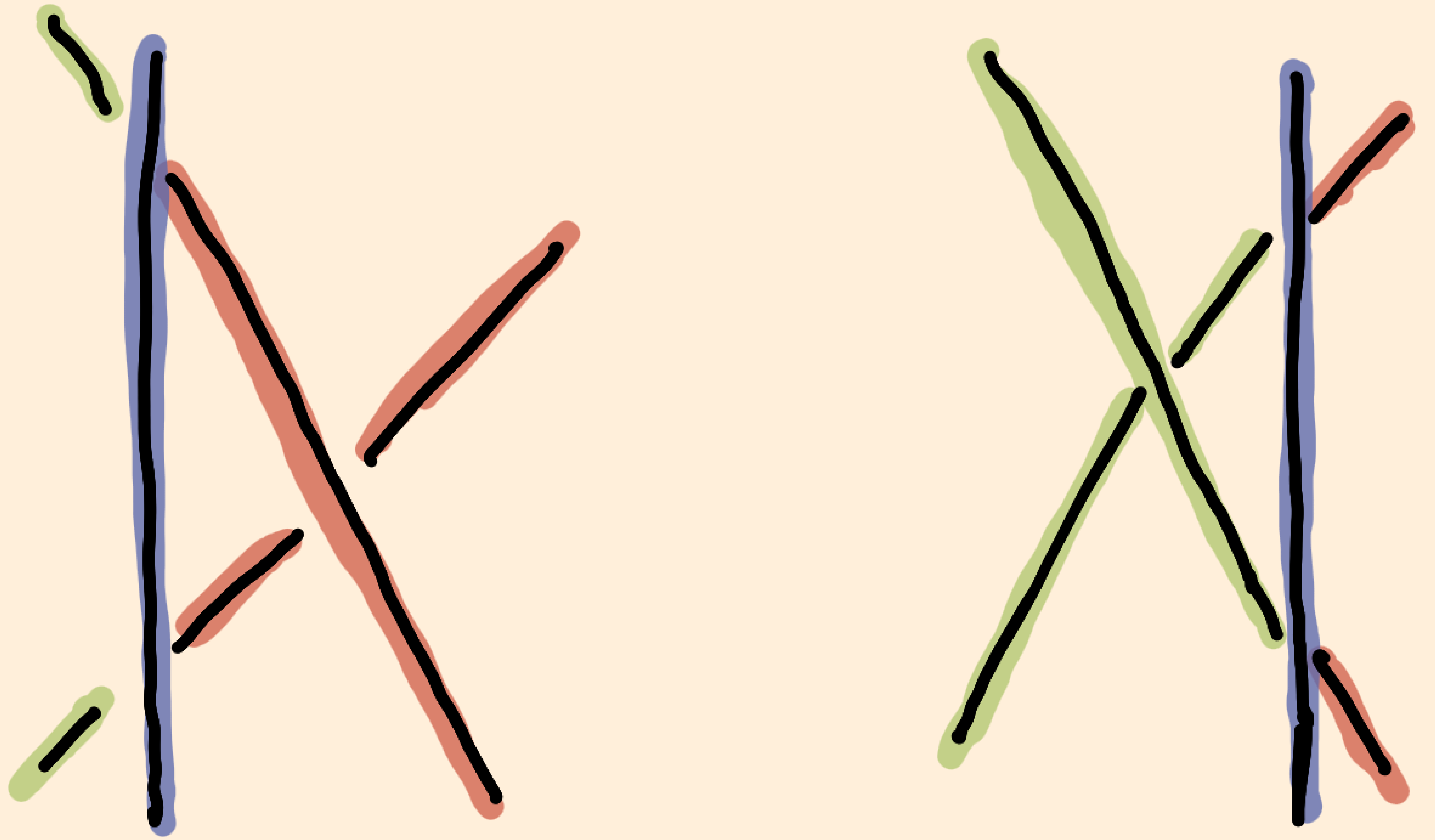
Proof: when we twist or untwist the knot in Reidemester I move, we can leave all the strands the same color.

Reidemeister II, either all strands are the same colour or the crossing creted or destroyed have 3 colours. Tricolourability is unaffected.

Similarly Reidemeister III move preserves tricolourability.



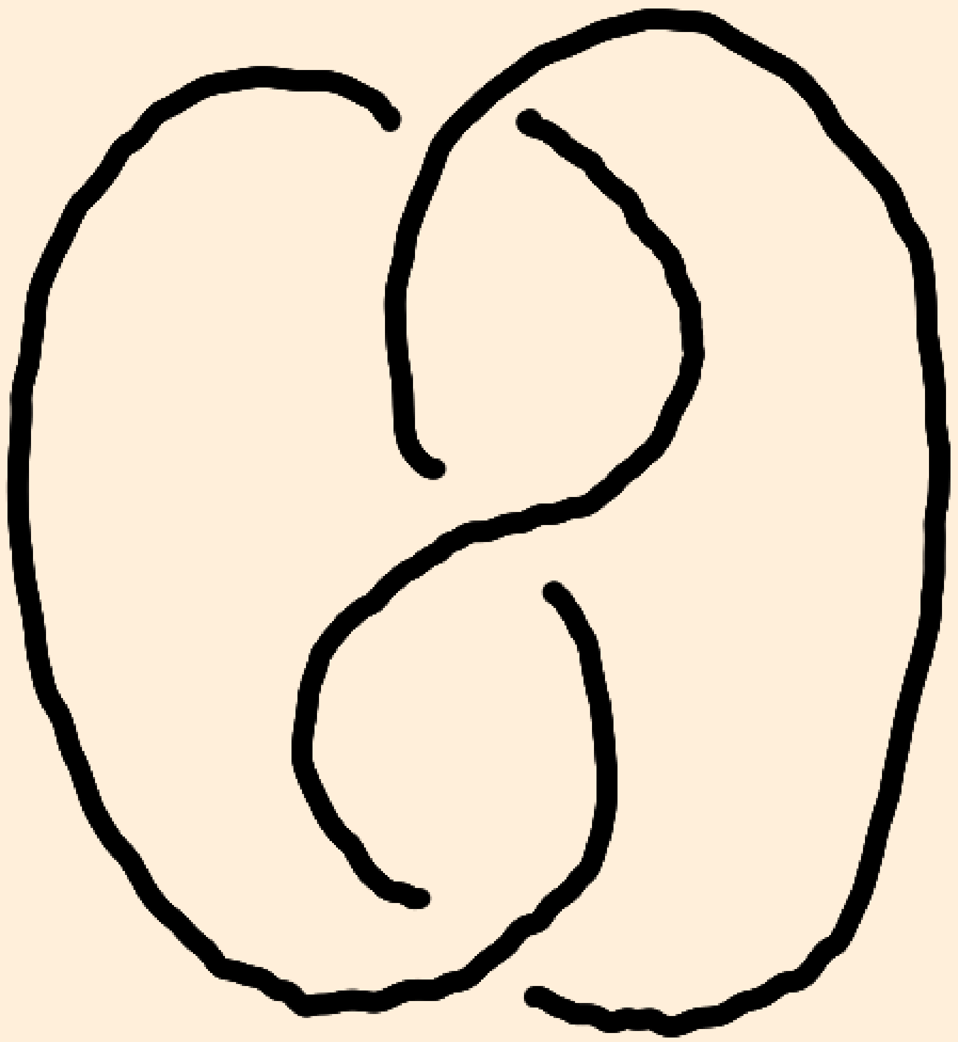


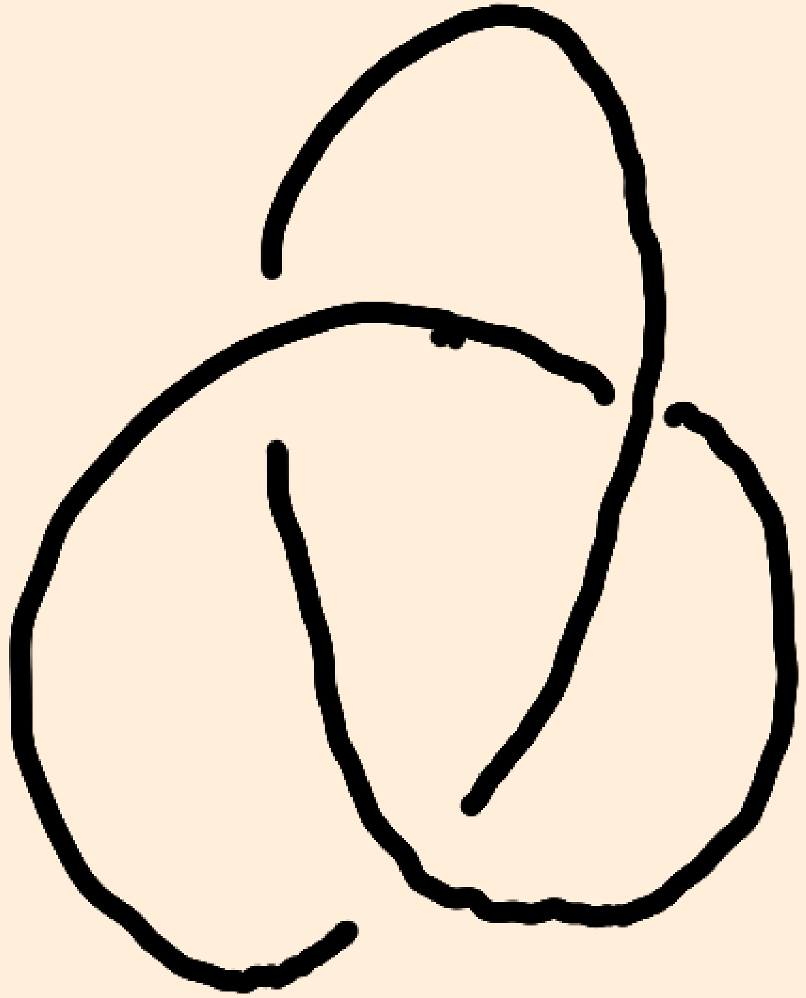


So, trefoil is different than unknot

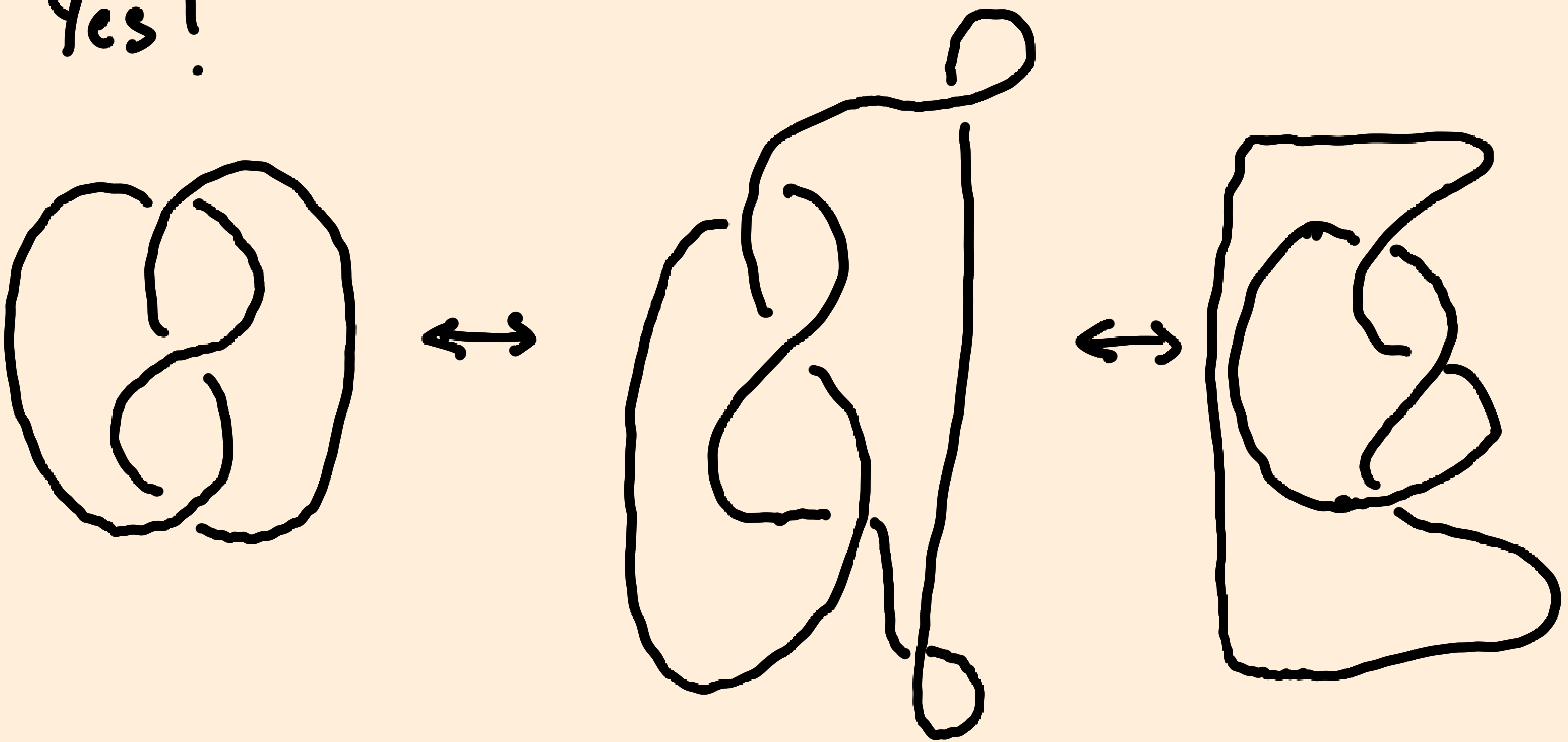
So, tricolourability is useful invariant!

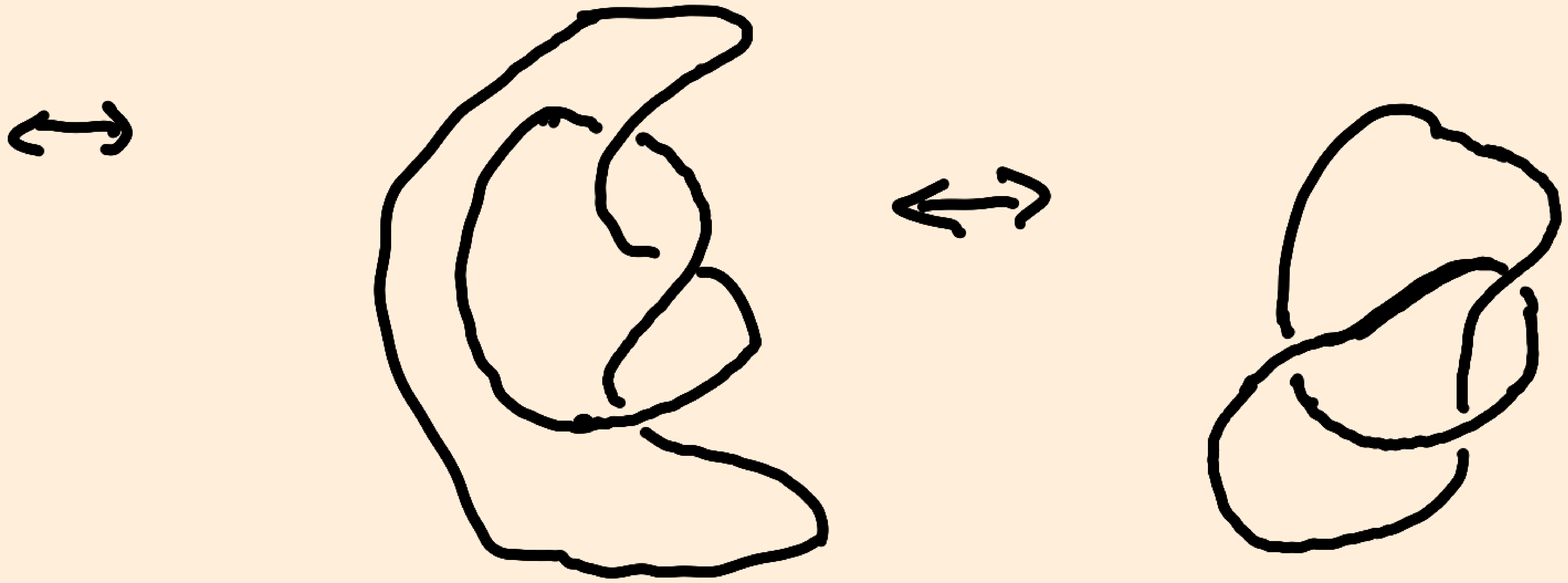
Are the following two knots equivalent?

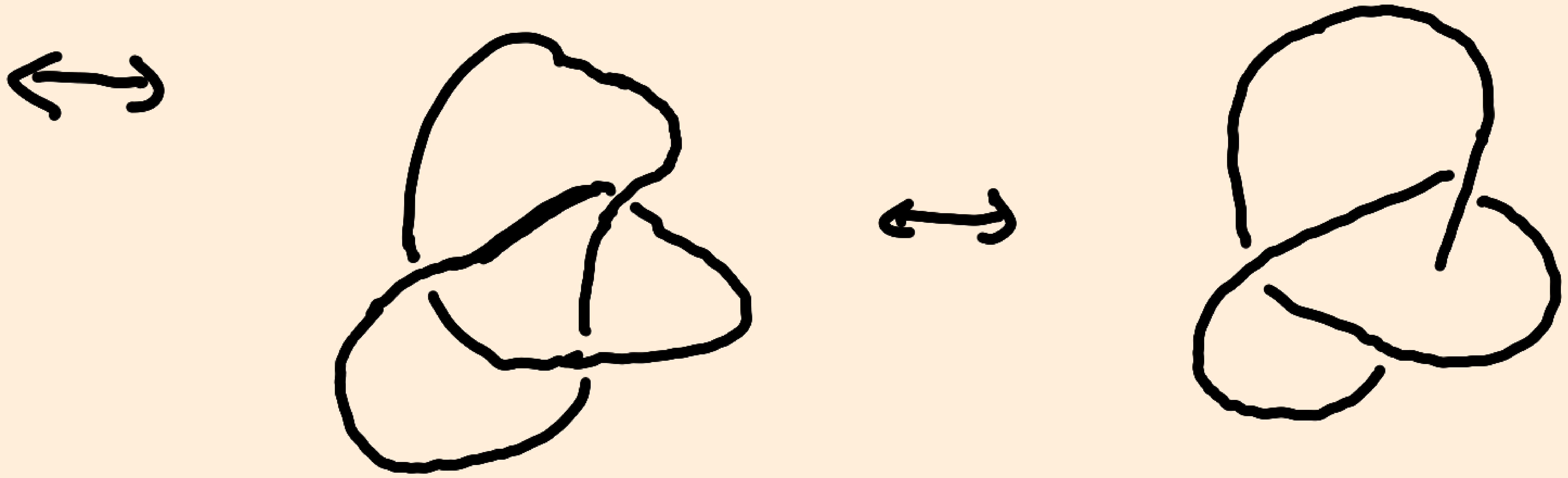


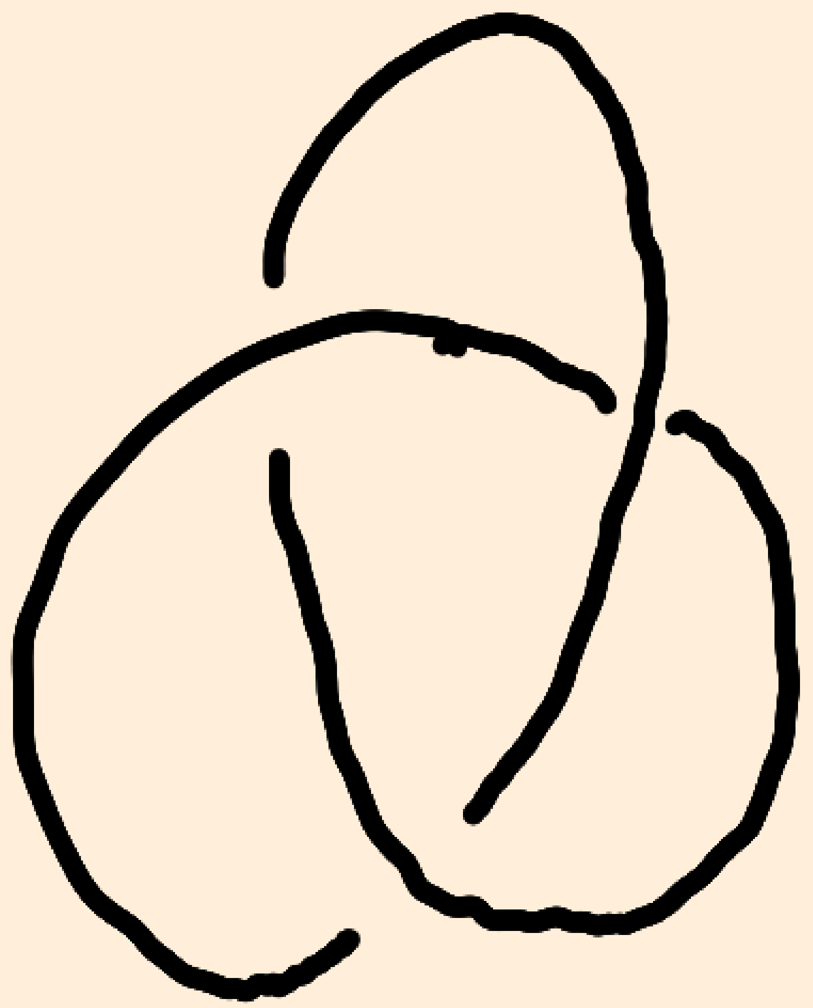


Yes!

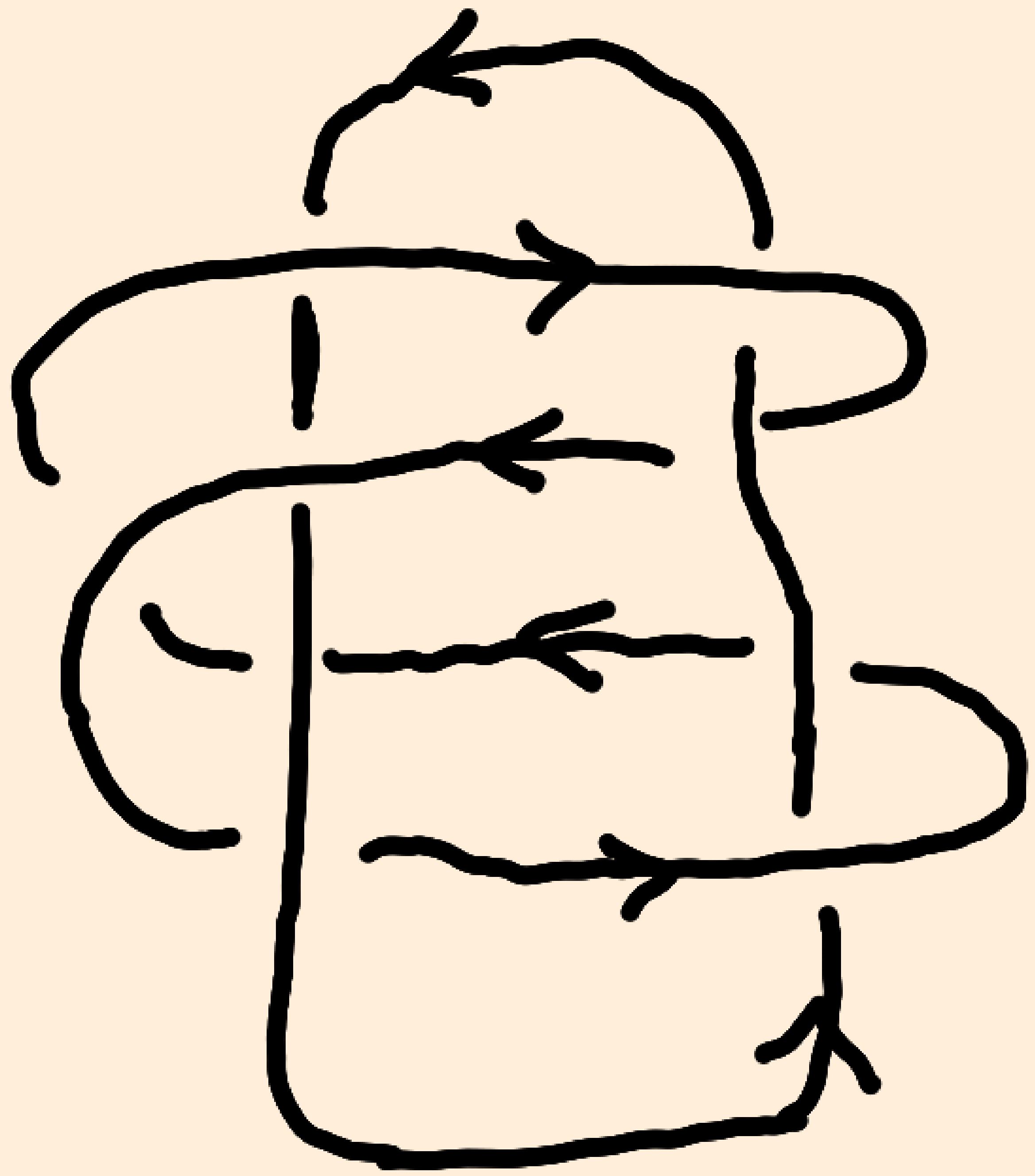


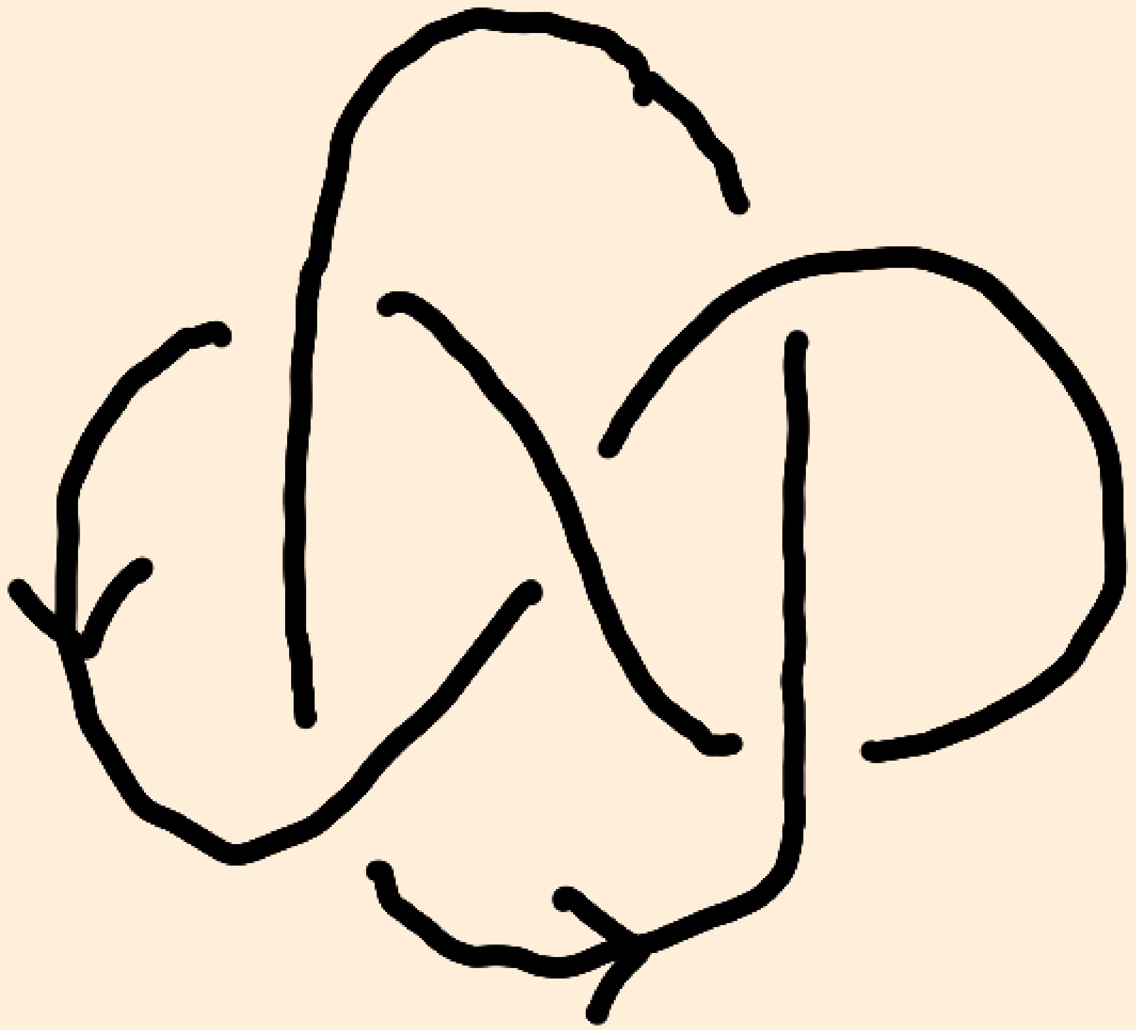




$\longleftrightarrow$ 

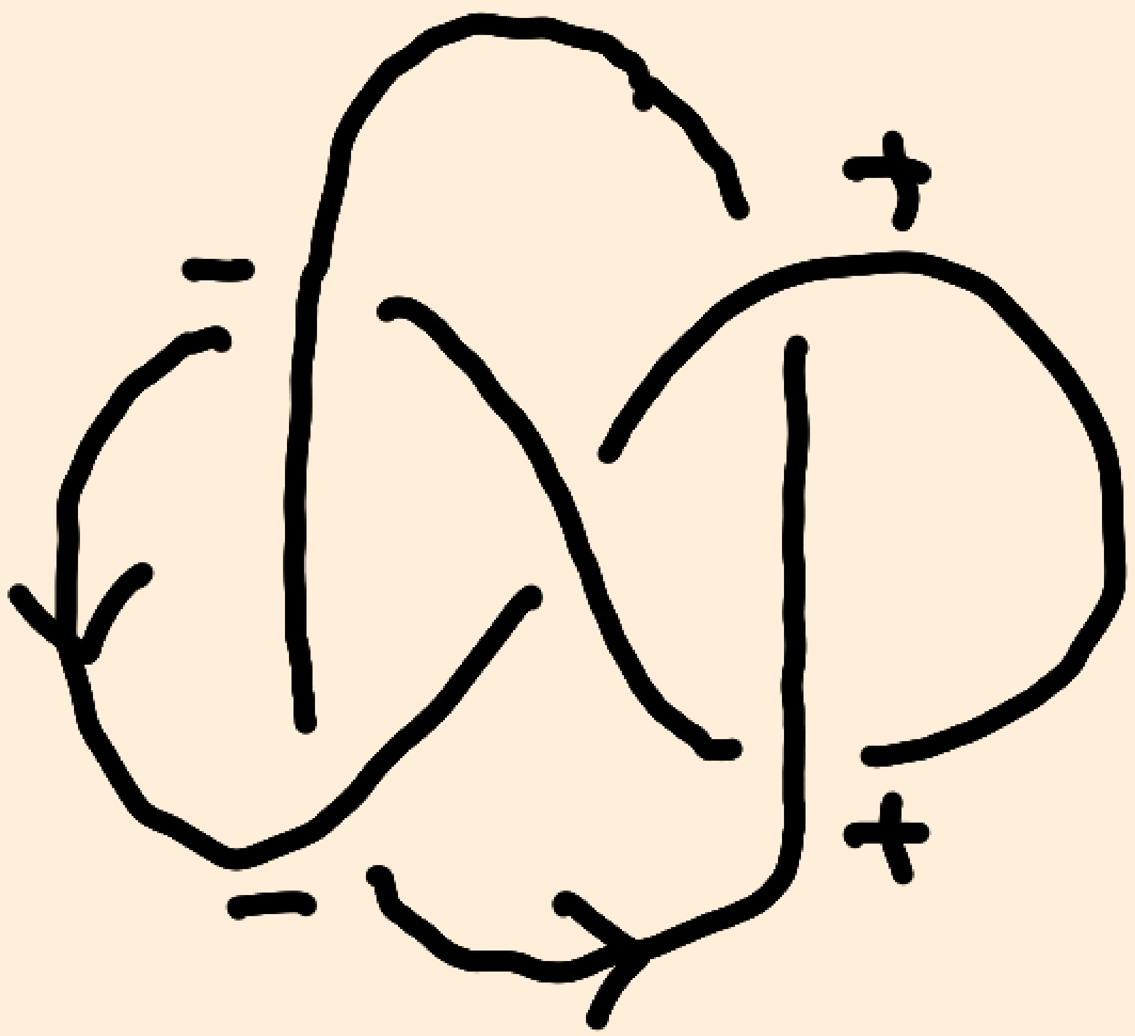
Are the following two equivalent?





Nope!

has linking number 4



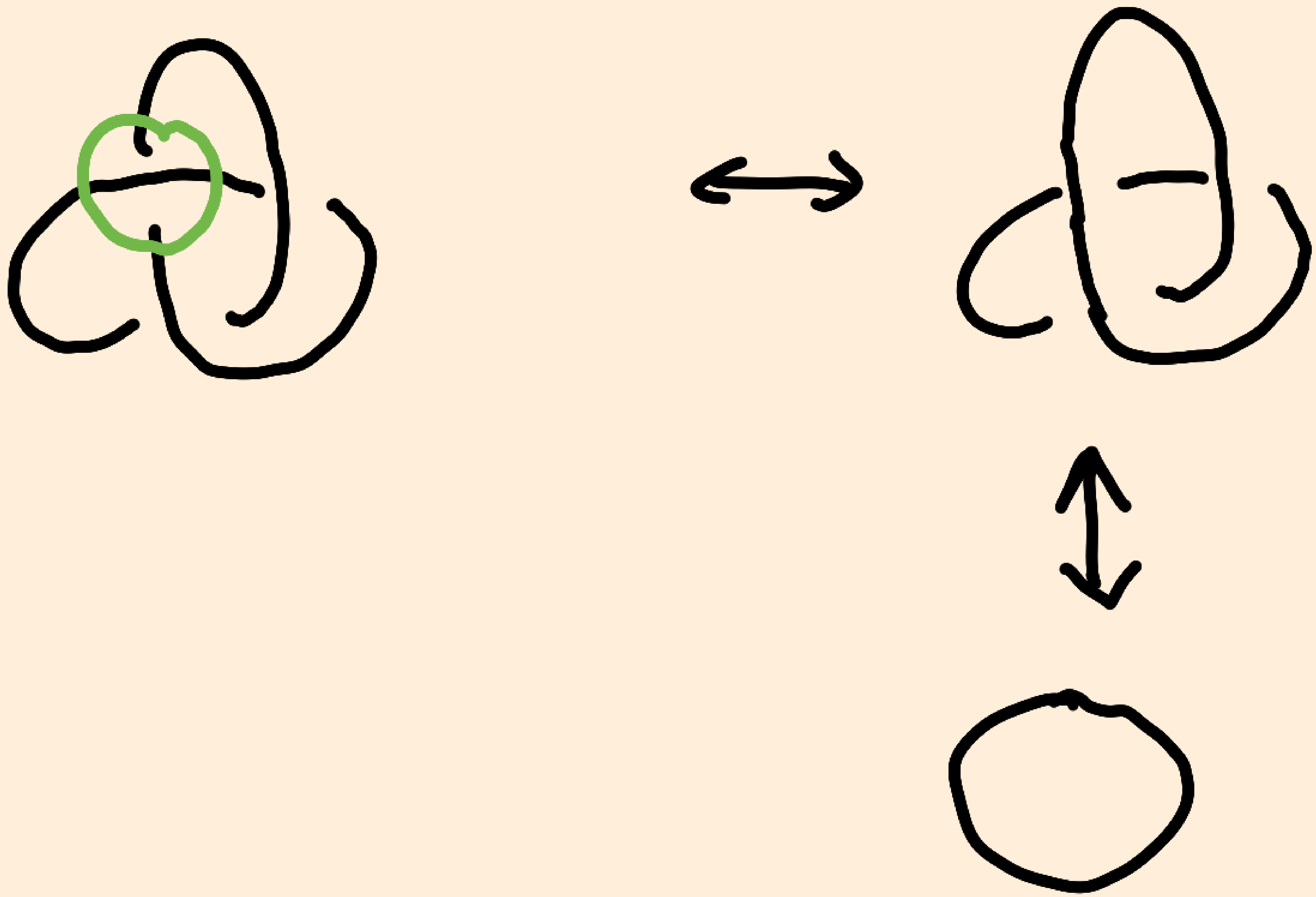
has linking number 0

More Invariances!

- unknotting number : the minimum number of crossings we need to exchange to get unknot.

What is unknoting number of trefoil?

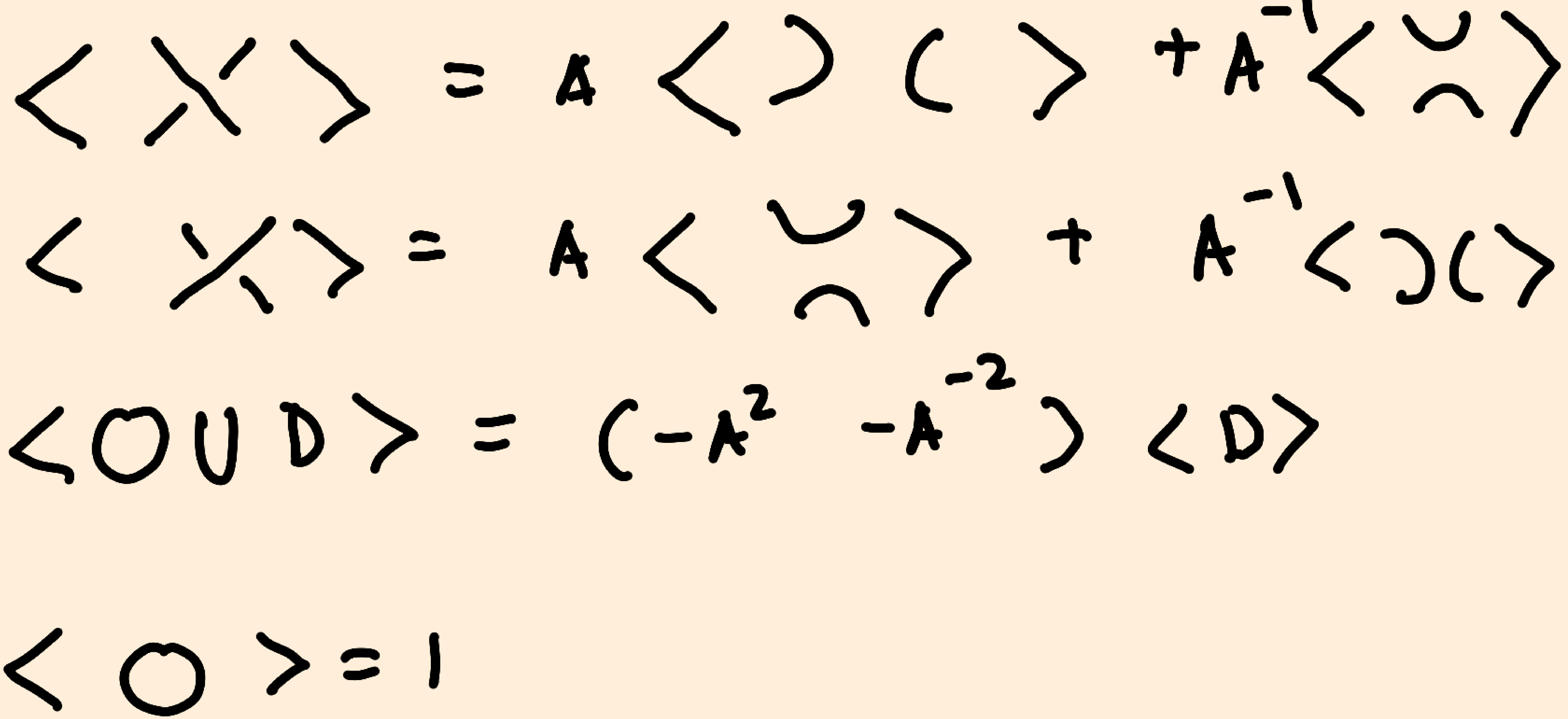
Example:



Bracket polynomial

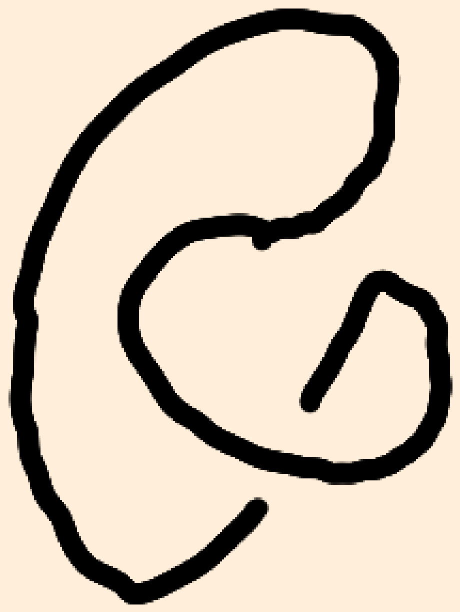
For a link diagram D in $\mathbb{R}^2$, the bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$.

We consider the following 4 formula:



NOTE:

$\langle$  $\rangle$ $=$ $A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

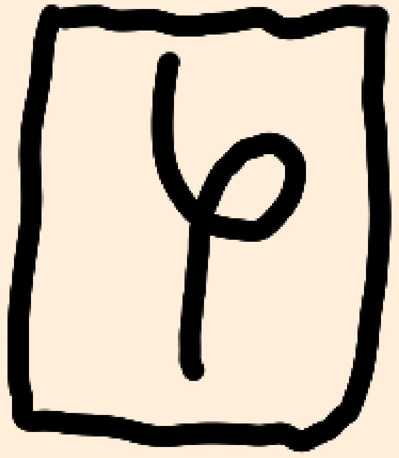
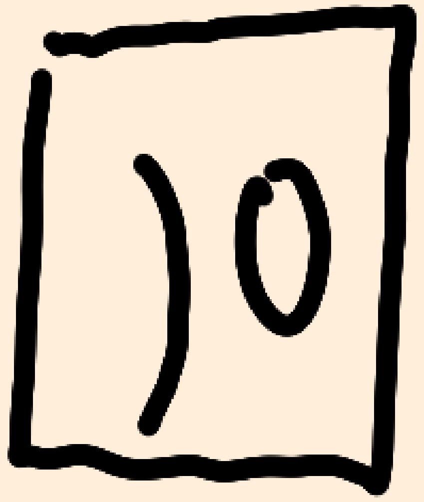
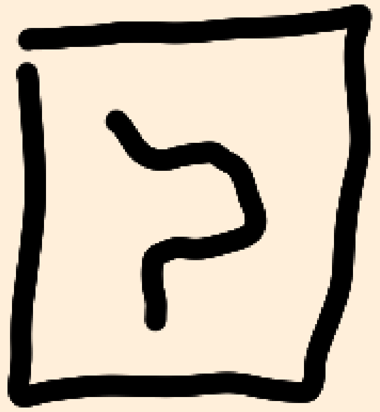
$= A^{2} \langle$  $\rangle$ $+$ $\langle$  $\rangle$

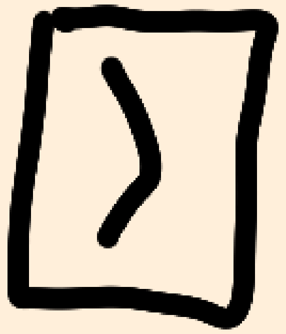
$A^{-1} \langle$  $\rangle$

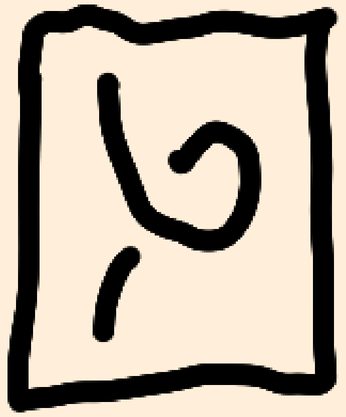
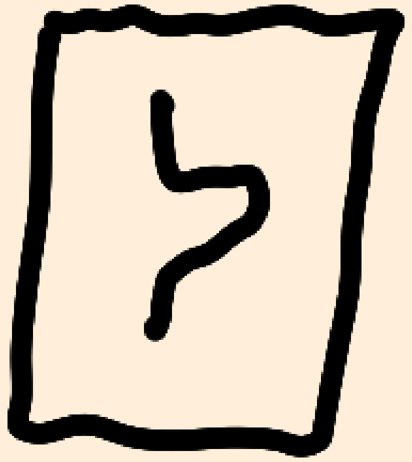
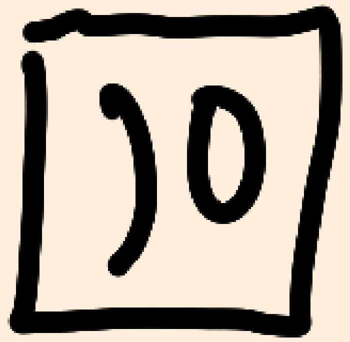
It can be shown that

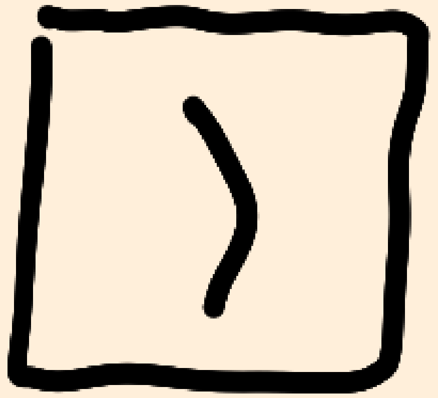
R_2 & R_3 does not change $\langle D \rangle$

but R_1 does.

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= -A^{3} \langle$  $\rangle$

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= -A^{3} \langle$  $\rangle$

How should we salvage this?

Definition :

For an oriented diagram D,
we define Writhe

$$w(D) = \#\text{ positive crossing of } D - \#\text{ negative crossing of } D$$

Big theorem:

Let L be an oriented link and D an oriented diagram of L.

Then $(-A^3)^{-w(D)} \langle D \rangle$ is invariant under R_1, R_2, R_3.

Put $A = t^{-1/4}$

Define Jone's polynomial $V_L(t)$ of oriented link L as

$$V_L(t) = \left. \frac{(-A^3)^{-w(D)}}{(-A^2 - A^{-2})} \langle D \rangle \right|_{A = t^{-1/4}}$$

Thankyou!

(Any Questions ?)